Analysis on graph-like spaces

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International Summer School on
Mathematics for the Micro/Nano-World:
Samarkand (Uzbekistan) September 2023

1. First lecture: fixing the notation — discrete and metric graphs
2. Second lecture: Laplacians on metric graphs
3. Third lecture: Spectra of compact metric graphs
4. Fourth lecture: Graph-like spaces: thin branched manifolds
5. Fifth lecture: Convergence Laplacians on thin branched manifolds

Motivation — The names and all that

Graph-like structure: a model for a quantum mechanical system of wires on which electrons travel

Models:

- discrete graphs and their Laplacians (difference operators on the vertices)
- metric graphs and their Laplacians, i.e., differential operators on the edges considered as intervals), a metric graph with such a (self-adjoint) operator is also called quantum graph
- thin branched structure (manifold) (or thick/fat graphs) and their Laplacians: an ε-neighbourhood X_ε of a metric graph X_0 embedded in $\mathbb{R}^2$ (or some other space) or a similarly defined space X_ε shrinking to X_0 as $\varepsilon \to 0$ together with a suitable Laplacian (e.g. Neumann or Dirichlet boundary condition on ∂X_ε)

Why thin branched structures are interesting?

- Modelling waves in thin branching “graph-like” structures: narrow waveguides, quantum wires, photonic crystals, blood vessels, lungs. Applications in nanotechnology, optics, chemistry, medicine
- Quantum graphs are systems of ODEs and therefore simpler objects than thin branched structures with their Laplacians (partial differential operators)
- If we can describe such a thin branched structure by a quantum graph, we have a good (sometimes even analytically “solvable”) model
- Quantum graphs themselves are sometimes considered as good models of electron spectra of molecules or Carbon nanostructures (graphene, nanotubes)
- discrete graph models sometimes also work well (tight binding model in Physics)
- transport of a wave of a certain frequency/energy is allowed if the frequency lies in the spectrum of an associated operator

Discrete graphs — fixing the notation

Definition

$G = (V, E, \partial, \bar{\cdot})$ discrete graph (later: functions $f \colon V \longrightarrow \mathbb{C}$ on vertices)

- consists of set of vertices V;
- set of edges E (both at most countable)
- connection map

$$\partial \colon E \longrightarrow V \times V, \qquad e \mapsto (\partial_- e, \partial_+ e) \tag{1.1}$$

  associating to $e \in E$ its initial ($\partial_- e$) and terminal ($\partial_+ e$) vertex (orientation)
- inversion map $\bar{\cdot} \colon E \longrightarrow E$, $e \mapsto \bar{e}$ with $\bar{\bar{e}} = e$ and $\partial_+ \bar{e} = \partial_e$
- $E_v := \{ e \in E \mid \partial_- e = v \}$ set of adjacent (outgoing) edges at $v \in V$
- $\deg v := |E_v|$ degree of vertex v

We allow loops ($\partial_- e = \partial_+ e$) or multiple edges (i.e., e_1, e_2) with $\partial_- e_1 = \partial_- e_2$ and $\partial_+ e_1 = \partial_+ e_2$
We allow infinite graphs, but we assume that the graph is locally finite, i.e., that $\deg v < \infty$ for all $v \in V$.

Discrete graphs — fixing the notation II

For a simple graph (without loops and multiple edges), we only need $G = (V, E)$ with $E \subset V \times V$:

- $e = (v, v') \in E \iff \bar{e} := (v', v) \in E$ (symmetric)
- $(v, v) \notin E$ for all $v \in V$ (no loops)
- connection map $\partial \colon E \longrightarrow V \times V$ is given by $e = (v, v') \mapsto (v', v)$ (injective)
- inversion map $\bar{\cdot} \colon E \longrightarrow E$, $\overline{(v', v)} = (v, v')$
- instead

$$\sum_{e \in E_v} (f(v) - f(\partial_+ e)) \quad \text{write} \quad \sum_{v' \sim v} (f(v) - f(v'))$$

for $f \colon V \longrightarrow \mathbb{C}$

Weighted discrete graphs

We often need edge and vertex weights (this allows us to consider various types of discrete Laplacians at the same time)

Definition

We call (G, γ, μ) a weighted discrete graph if

$$\begin{aligned} \gamma\colon E &\longrightarrow]0,\infty[, & e &\mapsto \gamma_e, \quad \gamma_{\bar e} = \gamma_e & &\text{(edge weight)} \\ \mu\colon V &\longrightarrow]0,\infty[, & v &\mapsto \mu(v) & &\text{(vertex weight)} \end{aligned}$$

($1/\gamma_e$ can be interpreted as a length of the edge e, see below).
For a discrete graph we have two natural weights:

- combinatorial weights: $\gamma_e = 1$ and $\mu(v) = 1$ for all e, v
- standard weights: $\gamma_e = 1$ and $\mu(v) = \deg v$
- electric network: $\mu(v) = 1$, $\gamma_e > 0$ conductance

Weighted degree, normalised weights

Definition

For an edge weight γ, we define the weighted degree $\deg_\gamma := \sum_{e \in E_v} \gamma_e$
A weighted graph is normalised if $\deg_\gamma(v) = \mu(v)$ for all $v \in V$.

Example

The standard weight ($\gamma_e = 1$, $\mu(v) = \deg(v)$) is normalised: $\deg_1(v) = \deg(v)$

We assume that

$$\varrho(v) := \frac{\deg_\gamma(v)}{\mu(v)} = \frac{1}{\mu(v)} \sum_{e \in E_v} \gamma_e$$

is uniformly bounded ($\sup_{v \in V} \varrho(v) < \infty$, only condition if $|V| = \infty$)

Example

- Standard or more generally normalised weights ($\varrho(v) = 1$)
- combinatorial weight: ($\gamma_e = 1$, $\mu(v) = 1$): ϱ bounded iff deg is uniformly bounded

Discrete Laplacians

Weighted Hilbert space:

$$\ell_2(V,\mu) := \{\, \varphi = (\varphi(v))_{v\in V} \mid \|\varphi\|^2_{\ell_2(V,\mu)} := \sum_{v\in V} |\varphi(v)|^2 \mu(v) < \infty \,\}.$$

Definition

The discrete Laplacian $\Delta = \Delta_{(G,\gamma,\mu)}$ associated to a weighted (discrete) graph (G,γ,μ) is defined as

$$(\Delta\varphi)(v) := \frac{1}{\mu(v)} \sum_{e\in E_v} \gamma_e(\varphi(v) - \varphi(\partial_+ e))$$

Exercise

- Check that $(\Delta\varphi)(v) = \varrho(v)\varphi(v) - \frac{1}{\mu(v)} \sum_{e\in E_v} \gamma_e \varphi(v_e)$
- Calculate Laplacian for combinatorial and standard weight
- Check that $\Delta \geq 0$ in $\ell_2(V,\mu)$, i.e., that $\langle \Delta\varphi, \varphi\rangle_{\ell_2(V,\mu)} \geq 0$ Hint: Try to express $\langle \Delta\varphi, \varphi\rangle_{\ell_2(V,\mu)}$ as a non-negative sum over $e \in E$.

Check in books, what the term "discrete Laplacian" actually means!

Metric graphs

Definition (metric graph)

Given a discrete graph $G = (V, E, \partial, \bar{\cdot})$ and a function $\ell \colon E \longrightarrow]0, \infty[$, $e \mapsto \ell_e$, ($\ell_{\bar{e}} = \ell_e$) edge length), we construct a metric space M as follows:

- for each edge $e \in E$ there is a "coordinate" map $\psi_e \colon [0, \ell_e] \longrightarrow M$ such that $M_e := \psi_e([0, \ell_e])$ is isometric with $[0, \ell_e]$
- We have $\psi_e(s) = \psi_{\bar{e}}(\ell_e - s)$, hence $M_{\bar{e}} = M_e$;
- M_e cover M $\quad (\bigcup_e M_e = M)$
- if $e, e' \in E_v$ then $\psi_e(0) = \psi_{e'}(0)$ (vertices are according to discrete structure)

M defined as above by the data (V, E, ∂, ℓ) is called a metric graph.

- In other words: We glue together copies of intervals $[0, \ell_e] \cong M_e$ according to the discrete graph
- Alternatively, you can think of M as a topological graph, where each edge $e \in E$ (topologically an interval) is associated a length $\ell_e > 0$.

In the sequel, we often identify $[0, \ell_e]$ with M_e and don't mention ψ_e.

Metric graphs II

- The graph M can be embedded $\mathbb{R}^d$; for $d = 2$, this may not always be possible;
- but note: embedding is unimportant for analysis of metric graphs (like Riemannian manifolds and submanifolds of $\mathbb{R}^d$), a metric graph M is uniquely defined by the data $G = (V, E, \partial, \bar{\cdot})$ and ℓ
- We also have a natural measure on M denoted by $\mathrm{d}s$, given by the sum of the Lebesgue measures $\mathrm{d}s_e$ on M_e (up to the boundary points, a null set).

To avoid technicalities, we assume

$$\ell_0 := \inf_{e \in E} \ell_e > 0. \tag{1.3}$$

Metric graphs: associated Hilbert space(s)

Associated Hilbert space is $\mathsf{L}_2(M)$ (with measure $\mathrm{d}s$).

- Using the coordinates ψ_e we set $f_e := f \circ \psi_e$
- as ψ_e are isometries and there are two orientations $e, \bar{e}$ (and points have measure 0), we have a natural identification of $f \in \mathsf{L}_2(M)$ with

$$(f_e)_{e\in E} \in \Big\{ (f_e)_e \in \bigoplus_{e\in E} \mathsf{L}_2(M_e) \,\Big|\, f_e(s) = f_{\bar{e}}(\ell_e - s)\ \forall e \in E, s \in [0,\ell_e] \Big\}$$

$$:= \bigoplus_{e\in E} \mathsf{L}_2(M_e)/\sim \quad (|E| = \infty\text{: } \bigoplus_e \text{ closure of algebraic direct sum})$$

- As norm we use (factor $1/2$ because edges come in two orientations)

$$\|f\|^2_{\mathsf{L}_2(M)} := \frac{1}{2} \sum_{e\in E} \int_{[0,\ell_e]} |f_e(s)|^2 \,\mathrm{d}s < \infty$$

Excursion: Sobolev spaces on intervals

Let us briefly review some facts:

- Let $I := [a, b]$ be a compact interval, $a < b$.

$$\mathsf{L}_2(I) := \left\{ f \colon I \longrightarrow \mathbb{C} \,\middle|\, \|f\|^2_{\mathsf{L}_2(I)} := \int_I |f(s)|^2 \, \mathrm{d}s < \infty \right\}$$

  (more precisely, f is a class of almost everywhere identical functions).
- f has a weak derivative $f' = h$ in $\mathsf{L}_2(I)$ if

$$\underbrace{\langle h, g \rangle_{\mathsf{L}_2(I)}}_{:= \int_I h(s)\overline{g(s)} \, \mathrm{d}s} = \langle f, -g' \rangle_{\mathsf{L}_2(I)}$$

  for all smooth g with support $\operatorname{supp} g := \overline{\{ s \in I \mid g(s) \neq 0 \}}$ inside $\mathring{I} = (a, b)$.
  (Take partial integration formula as definition for derivative)
- $\mathsf{H}^1(I) := \left\{ f \in \mathsf{L}_2(I) \,\middle|\, f' \in \mathsf{L}_2(I) \text{ weakly} \right\}$,
  $\|f\|^2_{\mathsf{H}^1(I)} := \|f\|^2_{\mathsf{L}_2(I)} + \|f'\|^2_{\mathsf{L}_2(I)}$

A simple Sobolev trace estimate

We have an important lemma, assuring that functions in $\mathrm{H}^1(I)$ are actually continuous and $f(b)$ makes sense for $f \in \mathrm{H}^1(I)$.
Its proof is rather simple so we include it here:

Lemma

1. *We have* $|f(s_1) - f(s_2)|^2 \le |s_1 - s_2| \|f'\|^2_{\mathrm{L}_2(I)}$ *for* $f \in \mathrm{H}^1(I)$. *In particular, functions in* f *are continuous, i.e.,* $\mathrm{H}^1(I) \subset \mathrm{C}(I)$ *(space of continuous functions on* I*).*[a]
2. *There exists* $C(\ell) > 0$ *(depending only on* $\ell := b - a > 0$*) such that*
$$|f(b)|^2 \le C(\ell) \|f\|^2_{\mathrm{H}^1(I)} \tag{1.4}$$
for all $f \in \mathrm{H}^1(I)$.

[a] More precisely, $f \in \mathrm{H}^1(I) \subset \mathrm{L}_2(I)$ is an equivalence class (a set) of functions being equal almost everywhere. But if there is a continuous representant, it is unique, since two different continuous functions must differ on on set of positive measures, hence the embedding $\mathrm{H}^1(I) \subset \mathrm{C}(I)$ is well-defined.

Proof of Soblev trace lemma

Proof.

(1) We have

$$f(s_2) - f(s_1) = \int_{s_1}^{s_2} f'(s)\,\mathrm{d}s, \tag{1.5}$$

hence by Cauchy-Schwarz

$$\begin{aligned} |f(s_2) - f(s_1)|^2 = \Big|\int_{s_1}^{s_2} 1 \cdot f'(s)\,\mathrm{d}s\Big|^2 &\le \int_{s_1}^{s_2} 1^2\,\mathrm{d}s \cdot \int_{s_1}^{s_2} |f'(s)|^2\,\mathrm{d}s \\ &\le |s_1 - s_2|\,\|f'\|^2_{\mathsf{L}_2(I)}. \end{aligned}$$

(2) Assume first that $f(a) = 0$, then by (1), we have

$$|f(b)|^2 \le \ell \int_a^b |f'(s)|^2\,\mathrm{d}s = \ell\|f'\|^2_{\mathsf{L}_2(I)}.$$

□

Proof of Soblev trace lemma II

Proof continued.

Now replace f by $\widetilde{f}(s) = \chi(s)f(s)$, where $\chi(a) = 0$ and $\chi(b) = 1$ (e.g. $\chi(s) = (s-a)/\ell$). Then

$$\widetilde{f}' = \chi f' + \chi' f, \qquad |\widetilde{f}'|^2 \le 2|f'|^2 + (2/\ell^2)|f|^2$$

and hence

$$|f(b)|^2 = |\widetilde{f}(b)|^2 \le \ell \|\widetilde{f}'\|^2_{\mathsf{L}_2(I)} \le 2\max\{\ell, \ell^{-1}\}\bigl(\|f'\|^2_{\mathsf{L}_2(I)} + \|f\|^2_{\mathsf{L}_2(I)}\bigr).$$

□

- The optimal constant is $C(\ell) = \coth(\ell/2) = \mathrm{O}(\ell^{-1})$ as $\ell \to 0$.
- Higher order spaces are defined recursively as

$$\mathsf{H}^k(I) := \bigl\{\, f \in \mathsf{H}^{k-1}(I) \,\big|\, f' \text{ exists weakly in } \mathsf{L}_2(I) \,\bigr\},$$

$$\|f\|^2_{\mathsf{H}^k(I)} := \sum_{j=0}^{k} \|f^{(j)}\|^2_{\mathsf{L}_2(I)}.$$

for $k \ge 1$. We set $\mathsf{H}^0(I) := \mathsf{L}_2(I)$. Moreover, it follows from Sobolev trace lemma that $f^{(j)}(s)$ is defined for $f \in \mathsf{H}^k(I)$ for all $0 \le j \le k-1$ and $s \in I$.

Sobolev spaces on metric graphs

Let M be a metric graph given by $(V, E, \partial, \bar{\cdot}, \ell)$.

Definition

We call

$$\mathsf{H}^k_{\mathrm{dec}}(M) := \bigoplus_{e \in E} \mathsf{H}^k(M_e)/\sim$$

the decoupled Sobolev space of order k on M, $\|f\|^2_{\mathsf{H}^k_{\mathrm{dec}}(M)} := \frac{1}{2} \sum_e \|f_e\|^2_{\mathsf{H}^k(M_e)}$

By the Sobolev trace lemma, $\mathsf{H}^1(M_e) \subset \mathsf{C}(M_e)$, hence it makes sense to define

$$\mathsf{H}^1(M) := \mathsf{H}^1_{\mathrm{dec}}(M) \cap \mathsf{C}(M), \tag{2.1}$$

i.e., a function $f \in \mathsf{H}^1_{\mathrm{dec}}(M)$ lies in $\mathsf{H}^1(M)$ iff

$$f_{e_1}(v) = f_{e_2}(v) \quad \text{for all } e_1, e_2 \in E_v \text{ and all } v \in V. \tag{2.2}$$

Here, we use the convention

$$f_e(v) := \begin{cases} f_e(0), & v = \partial_- e \\ f_e(\ell_e), & v = \partial_+ e, \end{cases} \tag{2.3}$$

the unoriented evaluation of f at v. Denote common value by $f(v)$.

Evaluation of functions at vertices

Lemma (later used to define standard Laplacian on metric graph)

Assume that $\ell_0 := \inf_{e \in E} \ell_e > 0$*, then the evaluation map*

$$\Gamma \colon \mathrm{H}^1(M) \longrightarrow \ell_2(V, \deg), \qquad f \mapsto (f(v))_{v \in V}$$

is bounded. Moreover, $\mathrm{H}^1(M)$ *is closed in* $\mathrm{H}^1_{\mathrm{dec}}(M)$*, hence itself a Hilbert space.*

Proof.

By the Sobolev trace lemma, we have $|f_e(0)|^2 \le C(\ell_e)\|f_e\|^2_{\mathrm{H}^1(M_e)}$ ($C(\ell_e) \sim 1/\ell_e$), hence

$$\begin{aligned}\|\Gamma f\|^2_{\ell_2(V,\deg)} &= \sum_{v \in V} |f(v)|^2 \deg v = \sum_{v \in V} \sum_{e \in E_v} |f(v)|^2 = \sum_{e \in E} |f_e(0)|^2 \\ &\le \sup_e C(\ell_e) \sum_{e \in E} \|f_e\|^2_{\mathrm{H}^1(M_e)} = 2 \sup_e C(\ell_e) \|f\|^2_{\mathrm{H}^1_{\mathrm{dec}}(M)}.\end{aligned}$$

Since $\ell_e \ge \ell_0 > 0$, we have $\sup_e C(\ell_e) < \infty$. □

Evaluation of functions at vertices II

Proof continued.

For $f \in \mathsf{H}^1_{\mathrm{dec}}(M)$ Consider

$$\Gamma_{\mathrm{dec}} \colon \mathsf{H}^1_{\mathrm{dec}}(M) \longrightarrow \mathscr{G}^{\mathrm{dec}} := \bigoplus_{v \in V} \mathbb{C}^{E_v}, \quad \Gamma_{\mathrm{dec}} f := (\underline{f}(v))_{v \in V}, \quad \underline{f}(v) := (f_e(0))_{e \in E_v}.$$

- By the same argument as above, it can be seen that Γ_{dec} is bounded.
- Now, embed $\ell_2(V, \deg)$ into $\mathscr{G}^{\mathrm{dec}}$ by setting $\varphi \mapsto (\varphi(v)(1, \dots, 1))_{v \in V}$, where $(1, \dots, 1) \in \mathbb{C}^{E_v}$ is the vector with all its $\deg v$ entries being 1;
- note that image of $\ell_2(V, \deg)$ is a closed subspace of $\mathscr{G}^{\mathrm{dec}}$.
- We can consider now $\mathsf{H}^1(M)$ as the preimage of a closed set (the image of $\ell_2(V, \deg)$ in $\mathscr{G}^{\mathrm{dec}}$) under a continuous mapping (Γ_{dec}), hence $\mathsf{H}^1(M)$ is closed, and hence itself a Hilbert space.

□

Evaluation of derivatives at vertices

If $f \in \mathrm{H}^2_{\mathrm{dec}}(M)$, we set

$$f_e'(v) := \begin{cases} -f_e'(0), & v = \partial_- e \\ f_e'(\ell_e), & v = \partial_+ e, \end{cases} \tag{2.4}$$

the oriented evaluation of f_e' at v.

Remark

- The choice of sign is guided by the formula $\int_0^{\ell_e} g''(s)\,\mathrm{d}s = [g'(s)]_0^{\ell_e} = g'(\ell_e) - g'(0)$.
- Think of the derivative as a vector field being evaluated with respect to the orientation, while f is a scalar function evaluated without orientation.
- The value $f_e'(v)$ is the derivative towards the vertex v.
- Be aware of the fact that others may use the opposite convention!

Excursion: Sesquilinear and quadratic forms and associated operators

Definition

- $\mathscr{H}$ Hilbert space,
- $\mathscr{D} \subset \mathscr{H}$ a linear subspace
- $\mathfrak{d}\colon \mathscr{D} \times \mathscr{D} \longrightarrow \mathbb{C}$ sesquilinear form (linear in the first, antilinear in the second argument: $\mathfrak{d}(f, \lambda g) = \overline{\lambda}\mathfrak{d}(f, g)$)
- $\mathfrak{d}$ positive ($\mathfrak{d}(f, f) \geq 0$ for all $f \in \mathscr{D}$)
- set $\operatorname{dom} \mathfrak{d} := \mathscr{D}$ (domain of the (quadratic) form $\mathfrak{d}$)
- We always assume that $\mathscr{D}$ is dense in $\mathscr{H}$.

Given a sesquilinear form, its associated quadratic form is given by

$$\mathfrak{d}(f) := \mathfrak{d}(f, f) \text{ } (\geq 0 \text{ in our case}).$$

Note that given a quadratic form, its associated sesquilinear form can be recovered by

$$\mathfrak{d}(f, g) := \frac{1}{4} \sum_{k=0}^{3} \mathrm{i}^k \mathfrak{d}(f + \mathrm{i}^k g).$$

Excursion: Sesquilinear and quadratic forms and associated operators II

The domain of $\mathfrak{d}$ carries a natural norm given by

$$\|f\|_{\mathfrak{d}}^2 := \|f\|_{\mathscr{H}}^2 + \mathfrak{d}(f). \tag{2.6}$$

Definition

We say that $\mathfrak{d}$ is a closed (quadratic) form if $(\operatorname{dom}\mathfrak{d}, \|\cdot\|_{\mathfrak{d}})$ is complete, i.e., itself a Hilbert space.

Theorem (see e.g. [Kat66, Thm. VI.2.1])

Let $\mathfrak{d}$ be a closed, positive quadratic form with domain $\operatorname{dom}\mathfrak{d}$ being dense in a Hilbert space $\mathscr{H}$. Then there is a unique self-adjoint and positive operator $\Delta \geq 0$ in $\mathscr{H}$ such that

$$\operatorname{dom}\Delta = \big\{\, f \in \operatorname{dom}\mathfrak{d} \,\big|\, \exists h \in \mathscr{H}\ \forall g \in \operatorname{dom}\mathfrak{d}\colon\ \mathfrak{d}(f,g) = \langle h, g\rangle_{\mathscr{H}} \,\big\}.$$

Moreover, $\Delta f = h$ is uniquely determined.

Typically, $\mathfrak{d}(f,g) = \langle h, g\rangle$ means to perform some sort of partial integration

Standard Laplacian on a metric graph

Set now

$$\mathfrak{d}(f) := \|f'\|^2_{\mathsf{L}_2(M)} = \frac{1}{2} \sum_{e \in E} \|f'_e\|^2_{\mathsf{L}_2(M_e)}, \qquad \operatorname{dom} \mathfrak{d} := \mathsf{H}^1(M). \tag{2.7}$$

Theorem

The quadratic form $\mathfrak{d}$ is positive and closed. The associated operator, denoted by Δ_M is given by

$$\operatorname{dom} \Delta_M = \Big\{ f \in \mathsf{H}^2_{\mathrm{dec}}(M) \,\Big|\, f \text{ continuous, } \forall v \in V\colon \sum_{e \in E_v} f'_e(v) = 0 \Big\}. \tag{2.8}$$

Proof.

That $\mathfrak{d}$ is a closed quadratic form is nothing but the fact that $\mathsf{H}^1(M)$ with norm given by $\|f\|^2_{\mathsf{H}^1(M)} = \mathfrak{d}(f) + \|f\|^2_{\mathsf{L}_2(M)}$ is complete, i.e., a Hilbert space (see Lemma 10). □

Standard Laplacian on a metric graph II

Proof continued.

For the associated operator, we calculate (provided f_e is regular enough to do partial integration: $f_e \in \mathrm{H}^2(M_e)$)

$$\mathfrak{d}(f,g) = \frac{1}{2}\sum_{e \in E} \int_0^{\ell_e} f_e' \overline{g}_e' \,\mathrm{d}s = \frac{1}{2}\sum_{e \in E} \Big(\int_0^{\ell_e} (-f_e'')\overline{g}_e \,\mathrm{d}s + \big[f_e'\overline{g}_e\big]_0^{\ell_e} \Big),$$

functions g vanishing near boundary (dense in $\mathrm{L}_2(M)$!), $\rightsquigarrow$ $(\Delta f)_e = -\frac{1}{2} f_e''$ (or $\Delta f = -f''$ on M)

For the boundary term, we have

$$\frac{1}{2}\sum_{e \in E} \big[f_e'\overline{g}_e\big]_0^{\ell_e} = \sum_{v \in V} f_e'(v)\overline{g}(v)$$

(Recall our sign definition for $f_e'(v)$ in (2.4)! Recall also that $g_e(v) = g(v)$ is independent of $e \in E_v$.) Reordering of the sum gives

$$\sum_{e \in E} f_e'(v)\overline{g}(v) = \sum_{v \in V} \Big(\sum_{e \in E_v} f_e'(v) \Big) \overline{g}(v).$$

Since $g(v)$ can be arbitary for $g \in \mathrm{H}^1(M)$ we must have $\sum_{e \in E_v} f_e'(v) = 0$. □

More Laplacians on a metric graph

- Due to $\sum_{e \in E_v} f_e'(v) = 0$ the Laplacian is also called Kirchhoff Laplacian.
- We have many other possibilities to define self-adjoint operators acting as $-f''$ on each edge: Assume that

$$\mathfrak{d}_q(f) := \|f'\|^2_{\mathsf{L}_2(M)} + \sum_{v \in V} q(v)|f(v)|^2, \quad \operatorname{dom} \mathfrak{d}_q = \mathsf{H}^1(M)$$

for a bounded function $q \colon V \longrightarrow \mathbb{R}$. As before, it is not difficult to see that the associated operator $\Delta_{(M,q)}$ acts as $\Delta_{(M,q)} f = -f''$ on the domain

$$\operatorname{dom} \Delta_{(M,q)} = \Big\{\, f \in \mathsf{H}^2_{\mathrm{dec}}(M) \,\Big|\, f \text{ continuous, } \forall v \in V \colon \sum_{e \in E_v} f_e'(v) + q(v) f(v) = 0 \,\Big\}. \tag{2.9}$$

More Laplacians on a metric graph II

General vertex conditions:

- choose linear subspace $\mathscr{G}_v \subset \mathbb{C}^{E_v}$, $\quad \mathscr{G} := \bigoplus_v \mathscr{G}_v$
  (so-called vertex space, allowed values of $\underline{f}(v) = (f_e(v)_{e \in E_v})$),
- set

$$\mathsf{H}^1_{\mathscr{G}}(M) := \{\, f \in \mathsf{H}^1_{\mathrm{dec}}(M) \,|\, \forall v \in V \colon\; \underline{f}(v) \in \mathscr{G}_v \,\}$$

$$\mathfrak{d}_{(\mathscr{G},Q)}(f) := \|f'\|^2_{\mathsf{L}_2(M)} + \sum_{v \in V} \langle Q(v)\underline{f}(v), \underline{f}(v)\rangle_{\mathscr{G}_v}, \quad \mathrm{dom}\,\mathfrak{d}_{(\mathscr{G},Q)} = \mathsf{H}^1_{\mathscr{G}}(M)$$

for linear $Q(v) \colon \mathscr{G}_v \longrightarrow \mathscr{G}_v$, $\sup_v \|Q(v)\| < \infty$. The associated operator $\Delta_{(M,\mathscr{G},Q)}$ acts as $\Delta_{(M,\mathscr{G},Q)} f = -f''$ on the domain

$$\mathrm{dom}\,\Delta_{(M,\mathscr{G},Q)} = \Big\{\, f \in \mathsf{H}^2_{\mathrm{dec}}(M) \,\Big|\, \forall v \in V \colon\; \underline{f}(v) \in \mathscr{G}_v,\; P_v \underline{f}'(v) + Q(v)\underline{f}(v) = 0 \,\Big\}.$$

(P_v is the projection onto $\mathscr{G}_v \subset \mathbb{C}^{E_v}$).

Other (trivial) Laplacians on a metric graph

Other trivial possibilities (extreme cases):

- decoupled Dirichlet Laplacian: fix $\mathscr{G}_v = \{0\}$, i.e., $f(v) = 0$ for all v, form is

$$\operatorname{dom} \mathfrak{d}^{D,\mathrm{dec}}(f) = \{ f \in \mathsf{H}^1(M) \mid f(v) = 0 \ \forall v \in V \}, \qquad \mathfrak{d}^{D,\mathrm{dec}}(f) = \|f'\|^2,$$

  the associated operator is $\Delta_M^{D,\mathrm{dec}} := \bigoplus_{e \in E} \Delta_{M_e}^{D} /\sim$

- decoupled Neumann Laplacian: fix $\mathscr{G}_v = \mathbb{C}^{E_v}$ then the quadratic form

$$\operatorname{dom} \mathfrak{d}^{N,\mathrm{dec}}(f) = \mathsf{H}^1_{\mathrm{dec}}(M), \qquad \mathfrak{d}^{N,\mathrm{dec}}(f) = \|f'\|^2,$$

  the associated operator is $\Delta_M^{N,\mathrm{dec}} := \bigoplus_{e \in E} \Delta_{M_e}^{N} /\sim$

Why are these operators called "decoupled"?

- the Laplacian (and its functions such as the heat or wave operator) are direct sum of the indivisual ones on each interval with Dirichlet resp. Neumann conditions
- In particular, no wave can travel through a vertex, and therefore cannot see any structure of the graph $\rightsquigarrow$ boring for applications!
- Nevertheless, both are useful as extreme case in proofs etc.

Excursion: Order on quadratic forms

Let $\mathfrak{d}_1$, $\mathfrak{d}_2$ be two positive closed quadratic forms in a Hilbert space $\mathscr{H}$.

Definition

We say that $\mathfrak{d}_1 \le \mathfrak{d}_2$ iff (see e.g. [Dav95, Sec. 4.4])

E. B. Davies, Spectral theory and differential operators, Cambridge University Press, Cambridge, 1995.

$$\operatorname{dom} \mathfrak{d}_1 \supset \operatorname{dom} \mathfrak{d}_2, \qquad \mathfrak{d}_1(f) \le \mathfrak{d}_2(f) \quad \forall f \in \operatorname{dom} \mathfrak{d}_2.$$

It follows that the resolvents of the associated operators Δ_1 and Δ_2 fulfil $(\Delta_2 + 1)^{-1} \le (\Delta_1 + 1)^{-1}$, and that by the min-max principle, we have $\lambda_k(\Delta_1) \le \lambda_k(\Delta_2)$ for the k-th eigenvalue (ordered with respect to multiplicity).

Exercise

Set $\widetilde{\mathfrak{d}}_i(f) := \mathfrak{d}_i(f)$ if $f \in \operatorname{dom} \mathfrak{d}_i$ and $\widetilde{\mathfrak{d}}_i(f) = \infty$ otherwise. Interprete now the (pointwise) inequality $\widetilde{\mathfrak{d}}_1 \le \widetilde{\mathfrak{d}}_2$. (For details, see [Dav95, Sec. 4.4])

Spectra of compact metric graphs

On a compact metric graph M (i.e., the underlying discrete graph is finite), we have the following (same is true for $\Delta_{M,\mathscr{G},Q}$):

Proposition

If M is a compact metric graph, then the standard (Kirchhoff) Laplacian Δ_M has purely discrete spectrum.

Proof.

We have

$$\{\, f \in \mathsf{H}^1(M) \mid f(v) = 0 \;\forall v \in V \,\} \subset \mathsf{H}^1(M) \subset \mathsf{H}^1_{\mathrm{dec}}(M),$$

and since $\mathfrak{d}^{\mathrm{D,dec}}$, $\mathfrak{d}$ (the form on $\mathsf{H}^1(M)$), $\mathfrak{d}^{\mathrm{N,dec}}$ all have the same action $\|f'\|^2$, we have the opposite inequality for the quadratic forms

$$\mathfrak{d}^{\mathrm{D,dec}} \ge \mathfrak{d} \ge \mathfrak{d}^{\mathrm{N,dec}}.$$

By the above excursion on order of quadratic forms, it follows

$$(0 \le)\,(\Delta_M^{\mathrm{D,dec}} + 1)^{-1} \le (\Delta_M + 1)^{-1} \le (\Delta_M^{\mathrm{N,dec}} + 1)^{-1} = \bigoplus_{e \in E} (\Delta_{M_e}^{\mathrm{N}} + 1)^{-1} / \sim$$

Since the underlying graph is finite and M_e compact, the RHS is a compact operator, hence also the LHS. □

Spectra of compact metric graphs II

- We also can conclude the eigenvalue estimates

$$\lambda_k(\Delta_M^{\mathrm{D,dec}}) \geq \lambda_k(\Delta_M) \geq \lambda_k(\Delta_M^{\mathrm{N,dec}})$$

  from the last proof, leading e.g. to a simple proof for the Weyl estimate for metric graph Laplacians.
- A similar result holds for more general vertex conditions (the Dirichlet and Neumann decoupled operators are extremal elements in a certain subclass of vertex couplings).

Calculation of the spectra

Let us now calculate the spectrum of the standard Laplacian Δ_M of a compact metric graph M. Since the spectrum of Δ_M is discrete we are looking for $\lambda \geq 0$ such that there is a non-trivial solution of $\Delta_M f = \lambda f$, i.e.,

$$-f_e'' = \lambda f_e \; \forall e \in E, \qquad f \text{ cont.}, \qquad \text{and} \qquad \sum_{e \in E_v} f_e'(v) = 0 \; \forall v \in V.$$

The first equation leads us to the two fundamental solutions

$$\Phi_e^+(t) := \frac{\sin(\sqrt{\lambda} t)}{\sin(\sqrt{\lambda}\ell_e)} \qquad \text{and} \qquad \Phi_e^-(t) := \frac{\sin\bigl(\sqrt{\lambda}(\ell_e - t)\bigr)}{\sin(\sqrt{\lambda}\ell_e)}$$

solving

$$-f_e'' = \lambda f_e \quad \text{on} \quad [0, \ell_e], \quad \begin{cases} \Phi_e^+(0) = 0, \Phi_e^+(\ell_e) = 1, \\ \Phi_e^-(0) = 1, \Phi_e^-(\ell_e) = 0 \end{cases}$$

We have to exclude those values λ for which $\sin(\sqrt{\lambda}\ell_e) = 0$, i.e., $\sqrt{\lambda}\ell_e \notin \pi\mathbb{N}$ (the spectrum of the decoupled Dirichlet operator $\Delta_M^{\mathrm{D,dec}}$), which is given by

$$\Sigma^{\mathrm{D}} = \Sigma_\ell^{\mathrm{D}} := \sigma(\Delta_M^{\mathrm{D,dec}}) = \Bigl\{ \frac{k^2\pi^2}{\ell_e^2} \Bigm| k = 1, 2, \dots, \; e \in E \Bigr\}. \qquad (3.2)$$

Calculation of the spectra II

Let us now make the ansatz

$$f_e(t) = \varphi(\partial_- e)\Phi_e^-(t) + \varphi(\partial_+ e)\Phi_e^-(t),$$

where $\varphi\colon V \longrightarrow \mathbb{C}$ are coefficients. Note that since $\varphi(v)$ does not depend on the edge $e \in E_v$, the continuity condition for $f = (f_e)_{e \in E}$ is automatically fulfilled.
Let us now check the condition on the derivatives: It is an easy exercise (recall the sign convention for $f_e'(v)$ in (2.4)) that

$$f_e'(v) = \varphi(v)\sqrt{\lambda}\cot(\sqrt{\lambda}\ell_e) - \varphi(\partial_+ e)\frac{\sqrt{\lambda}}{\sin(\sqrt{\lambda}\ell_e)}.$$

Therefore, we have the following:

Proposition

Let M be a compact metric graph. Assume that $\lambda > 0$ and λ is not in the Dirichlet spectrum $\Sigma^{\mathrm{D}} = \sigma(\Delta_M^{\mathrm{D,dec}})$. Then $\lambda \in \sigma(\Delta_M)$ (the spectrum of the standard Laplacian on M) iff there exists a non-trivial function $\varphi\colon V \longrightarrow \mathbb{C}$ such that

$$\sum_{e \in E} \frac{1}{\sin(\sqrt{\lambda}\ell_e)}\Big(\cos(\sqrt{\lambda}\ell_e)\varphi(v) - \varphi(\partial_+ e)\Big) = 0. \qquad (3.4)$$

Can be quite complicated if there is no simple relation between ℓ_e, $e \in E$

Equilateral metric graphs

A particular simple case if given when ℓ_e are all the same, say, $\ell_e = 1$.

Proposition

Let M be a compact metric graph with $\ell_e = 1$ for all $e \in E$ (equilateral metric graph).

- *If $\lambda \notin \Sigma^{\mathrm{D}} = \{ k^2\pi^2 \mid k = 1, 2, \dots \}$ then*

$$\lambda \in \sigma(\Delta_M) \qquad \textit{iff} \qquad \mu(\lambda) := 1 - \cos(\sqrt{\lambda}) \in \sigma(\Delta_{(G,\deg)}),$$

  ($\Delta_{(G,\deg)}$ standard (normalised) discrete Laplacian (weights $m(v) = \deg v$, $m_e = 1$)
- *Moreover, the multiplicity of an eigenvalue is preserved.*

Proof.

Recall that the discrete Laplacian is given by

$$(\Delta_{(G,\deg)}\varphi)(v) = \frac{1}{\deg v} \sum_{e \in E_v} \bigl(\varphi(v) - \varphi(\partial_+ e)\bigr) = \varphi(v) - \frac{1}{\deg v} \sum_{e \in E_v} \varphi(\partial_+ e).$$

□

Equilateral metric graphs

Proof continued.

If all lengths are the same, we can multiply (3.4) by $\sin\sqrt{\lambda}/(\deg v)(\neq 0)$ and end up with the equation

$$\cos(\sqrt{\lambda})\varphi(v) - \frac{1}{\deg v}\sum_{e\in E_v}\varphi(\partial_+ e) = 0,$$

which is equivalent with

$$(\Delta_{(G,\deg)}\varphi)(v) = (1-\cos(\sqrt{\lambda}))\varphi(v),$$

and the result follows. □

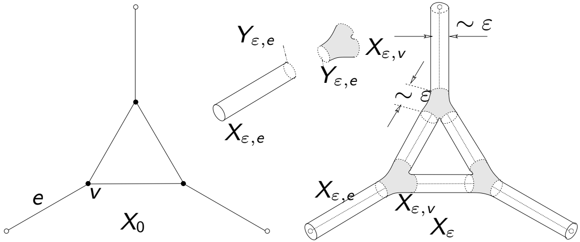
Remark

A discussion on the Dirichlet spectrum (the "exceptional" values $\sqrt{\lambda}\in\pi\mathbb{N}$) can be found in [LP08, Sec. 4–5] (see also the references therein). These eigenvalues are determined by the topology of the graph.

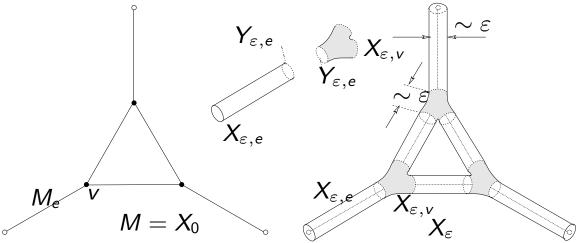
F. Lledó and O. Post, Eigenvalue bracketing for discrete and metric graphs, J. Math. Anal. Appl. **348** (2008), 806–833.

Thin branched manifolds

- Let M be a metric graph.
- Assume for simplicity that M is embedded in $\mathbb{R}^d$.
- Let X_ε be the $\varepsilon/2$-neighbourhood of M in $\mathbb{R}^d$ (possibly smoothened) near the vertices.
- There are other possibilites of defining spaces (manifolds) X_ε shrinking to $X_0 = M$ as $\varepsilon \to 0$.
- We will show that the Neumann Laplacian on X_ε converges to Δ_{X_0}.



Thin branched manifolds II



- We have a decomposition

$$X_\varepsilon = \bigcup_{e \in E} X_{\varepsilon,e} \cup \bigcup_{v \in V} X_{\varepsilon,v}, \tag{4.1}$$

where $X_{\varepsilon,e}$ and $X_{\varepsilon,v}$ are compact spaces with boundary, $(X_{\varepsilon,e})_{e \in E}$ and $(X_{\varepsilon,v})_{v \in V}$ are disjoint (up to measure 0 and $X_{\varepsilon,e} = X_{\varepsilon,\bar{e}}$)

- The edge neighbourhood $X_{\varepsilon,e}$ is isometric to a cylinder

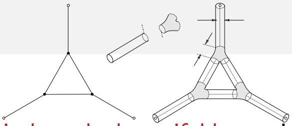
$$X_{\varepsilon,e} \cong M_e \times Y_{\varepsilon,e},$$

where $Y_{\varepsilon,e}$ is the transversal space (e.g. $Y_{\varepsilon,e} \cong B_{\varepsilon/2}(0)$), $Y_{\varepsilon,e} = \varepsilon Y_e$

- The vertex neighbourhood $X_{\varepsilon,v}$ is ε-homothetic to X_v, i.e.,

$$X_{\varepsilon,v} \cong \varepsilon X_v,$$

Thin branched manifolds III



Definition

The manifold X_ε with the above properties is called a thin branched manifold associated with the metric graph X_0.

Remark

- The manifold X_ε may have boundary or not. If X_ε has boundary, then also the transversal manifold Y_e has boundary.
- If we consider a graph M embedded in, say, $\mathbb{R}^2$ and if $\widetilde{X}_\varepsilon$ denotes its ε-neighbourhood, then we can define a similar decomposition as in (4.1), but the building blocks $\widetilde{X}_{\varepsilon,e}$ and $\widetilde{X}_{\varepsilon,v}$ are only approximatively isometric with $M_e \times \varepsilon Y_e$ and εX_v for some fixed Riemannian manifolds (Y_e, h_e) and (X_v, g_v). This may have two reasons:
  - We need a little space for the vertex neighbourhoods (of order ε), so that we need to replace the interval M_e by a slightly smaller one of length $\ell_e - O(\varepsilon)$.
  - The edges may be embedded as non-straight curves in $\mathbb{R}^2$. This leads to a slight deviation from the product metric.

  All these cases can be treated as a perturbation of the abstract situation above, see e.g. [Pos12, Sec. 5.4 and Sec. 6.7]).

O. Post, Spectral analysis on graph-like spaces, Lecture Notes in Mathematics, vol. 2039, Springer, Heidelberg, 2012.

Laplacians on thin branched manifolds

- Hilbert space is $\mathsf{L}_2(X_\varepsilon)$. In particular, we have ($m = d - 1$)

$$\|u\|^2_{\mathsf{L}_2(X)_\varepsilon} = \int_{X_\varepsilon} |u(x)|^2 \,\mathrm{d}x$$

$$= \varepsilon^m \frac{1}{2} \sum_{e \in E} \int_0^{\ell_e} \int_{Y_e} |u_e(s,y)|^2 \,\mathrm{d}y \,\mathrm{d}s + \varepsilon^{m+1} \sum_{v \in V} \int_{X_v} |u_v(x)|^2 \,\mathrm{d}x$$

  using the decomposition (4.1) and suitable identifications.
- As operator on X_ε, we consider the Laplacian with Neumann boundary conditions (if $\partial X_\varepsilon \neq \emptyset$). This operator can again be defined via a quadratic form, namely by

$$\mathfrak{d}_{X_\varepsilon}(u) := \|\nabla u\|^2_{\mathsf{L}_2(X_\varepsilon)}.$$

  Using again the decomposition (4.1), we have

$$\|\nabla u\|^2_{\mathsf{L}_2(X_\varepsilon)} = \int_{X_\varepsilon} |\nabla u|^2_{g_\varepsilon} \,\mathrm{d}x = \varepsilon^m \frac{1}{2} \sum_{e \in E} \int_0^{\ell_e} \int_{Y_e} \Big(|u_e'(s,y)|^2 + \frac{1}{\varepsilon^2} |\nabla_y u_e(s,y)|^2 \Big) \,\mathrm{d}y \,\mathrm{d}s$$

$$+ \varepsilon^{m-1} \sum_{v \in V} \int_{X_v} |\nabla u_v|^2 \,\mathrm{d}x$$

  denoting u_e' the derivative with respect to the longitudinal variable $s_e \in M_e$, and by ∇_y the derivative with respect to $y \in Y_e$.

Laplacians on thin branched manifolds

Let $\widetilde{u}_v(x) = u_v(x/\varepsilon)$ then (we later use no extra notation like $\widetilde{u}$)

- ("derivative is 1/length", change of variable)

$$|\nabla \widetilde{u}_v|^2 = \frac{1}{\varepsilon^2} |\nabla u_v|^2$$

- Moreover, we have the scaling behaviour

$$\|\widetilde{u}_v\|^2_{\mathsf{L}_2(X_{\varepsilon,v})} = \varepsilon^{m+1} \|u_v\|^2_{\mathsf{L}_2(X_v)} \qquad \text{and} \qquad \|\nabla u_v\|^2_{\mathsf{L}_2(X_{\varepsilon,v})} = \varepsilon^{m-1} \|\nabla u_v\|^2_{\mathsf{L}_2(X_v)}$$

- As domain for $\mathfrak{d}_{X_\varepsilon}$ we can use the completion of smooth functions on X with compact support (not necessarily away from the boundary) with respect to the norm

$$\|u\|^2_{\mathsf{H}^1(X_\varepsilon)} := \|u\|^2_{\mathsf{L}_2(X_\varepsilon)} + \|\nabla u\|^2_{\mathsf{L}_2(X_\varepsilon)}.$$

  It can be seen (using the first Gauss-Green formula) that the associated operator, denoted by Δ_{X_ε} is the usual Laplacian with Neumann boundary conditions $\partial_{\mathsf{n}} u = 0$ on ∂X_ε.

Excursion: Convergence of operators acting in different Hilbert spaces

How do define a convergence "$\Delta_{X_\varepsilon} \to \Delta_M$"?

- The operators act in different Hilbert spaces $\mathsf{L}_2(X_\varepsilon)$ and $\mathsf{L}_2(X_0)$
- No natural inclusion, In the lmit $X_0 = M$, there is a dimension reduction. We have $X_\varepsilon \to X_0$ (in the Gromov-Hausdorff sense)
- The operators are unbounded, so use their resolvents $R_\varepsilon := (\Delta_\varepsilon + 1)^{-1}$ $(\varepsilon \geq 0)$ (recall: Here, $\Delta_\varepsilon = \Delta_{X_\varepsilon} \geq 0$)

General concept: $\Delta_\varepsilon \geq 0$ self-adjoint operator in a Hilbert space $\mathscr{H}_\varepsilon$ $(\varepsilon \geq 0)$.
Set $R_\varepsilon := (\Delta_\varepsilon + 1)^{-1}$ for the resovent. Identify spaces $\mathscr{H}_\varepsilon$ and $\mathscr{H}_0$ via identification operators $J = J_\varepsilon \colon \mathscr{H}_0 \longrightarrow \mathscr{H}_\varepsilon$ ($\|J_\varepsilon\| \leq 1$, we suppress its ε-dependence):

Definition

We say that $J = J_\varepsilon$ is δ_ε-quasi unitary if

$$\|(\mathrm{id}_{\mathscr{H}_0} - J^* J) R_0\|_{\mathscr{H}_0 \to \mathscr{H}_0} \leq \delta_\varepsilon \qquad \text{and} \qquad \|(\mathrm{id}_{\mathscr{H}_\varepsilon} - JJ^*) R_\varepsilon\|_{\mathscr{H}_\varepsilon \to \mathscr{H}_\varepsilon} \leq \delta_\varepsilon. \tag{4.3}$$

We say that Δ_0 and Δ_ε are δ_ε-quasi unitarily equivalent if there is a δ_ε-quasi unitary operator J such that

$$\|JR_0 - R_\varepsilon J\|_{\mathscr{H}_0 \to \mathscr{H}_\varepsilon} \leq \delta_\varepsilon. \tag{4.4}$$

Excursion: Convergence of operators acting in different Hilbert spaces II

Definition (repetition)

We say that Δ_0 and Δ_ε are δ_ε-quasi unitarily equivalent if there is a δ_ε-quasi unitary operator J such that

$$\|(\mathrm{id}_{\mathscr{H}_0} - J^* J)R_0\| \le \delta_\varepsilon, \quad \|(\mathrm{id}_{\mathscr{H}_\varepsilon} - JJ^*)R_\varepsilon\| \le \delta_\varepsilon \quad \text{and} \quad \|JR_0 - R_\varepsilon J\| \le \delta_\varepsilon.$$

Remark

- Note that δ_ε-quasi unitarity is a quantitative generalisation of unitarity: if $\delta_\varepsilon = 0$, J is actually unitary.
- Moreover, δ_ε-quasi unitary equivalence is a quantitative generalisation of unitary equivalence: if $\delta_\varepsilon = 0$, then Δ_ε and Δ_0 are actually unitarily equivalent.

Exercise

- Show that $\|(\mathrm{id}_{\mathscr{H}_0} - J^* J)R_0\| \le \delta_\varepsilon$ iff $\|f - J^* Jf\| \le \delta_\varepsilon \|(H_0 + 1)f\|$ for all $f \in \operatorname{dom} H_0$
- There is a stronger version: Show that $\|R_0(\mathrm{id}_{\mathscr{H}_0} - J^* J)R_0\| \le \delta_\varepsilon$ iff

$$\|f\|^2 - \|Jf\|^2 \le \delta_\varepsilon \|(H_0 + 1)f\|^2 \quad \forall f \in \operatorname{dom} H_0$$

Excursion: Convergence of operators acting in different Hilbert spaces III

Definition

We say that Δ_ε converges to Δ_0 in the generalised norm resolvent sense ($\Delta_\varepsilon \overset{\text{gnrc}}{\to} \Delta_0$) if Δ_0 and Δ_ε are δ_ε-quasi unitarily equivalent for $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$ (convergence speed).

Theorem

Assume $\Delta_\varepsilon \overset{\text{gnrc}}{\to} \Delta_0$, then (for a proof see [Pos12, Ch. 4]):

- *Convergence of operator functions: We have*

$$\|\varphi(\Delta_\varepsilon)J - J\varphi(\Delta_0)\|_{\mathscr{H}_0 \to \mathscr{H}_\varepsilon} \le C_\varphi \delta_\varepsilon \quad \text{and} \quad \|\varphi(\Delta_\varepsilon) - J\varphi(\Delta_0)J^*\|_{\mathscr{H}_\varepsilon \to \mathscr{H}_\varepsilon} \le C'_\varphi \delta_\varepsilon$$

  for suitable φ and some universal constants $C_\varphi, C'_\varphi > 0$ depending only on φ. In particular, $\varphi(\lambda) = \mathrm{e}^{-t\lambda}$, $t > 0$ and $\varphi = \mathbb{1}_I$ with $\partial I \cap \sigma(\Delta_0) = \emptyset$).
- *Convergence of discrete spectrum: Let λ_0 be a simple eigenvaluewith eigenfunction φ_0, then, for each $\varepsilon > 0$ (small enough), there exists a simple discrete eigenvalue λ_ε with eigenfunction φ_ε of Δ_ε such that $\lambda_\varepsilon \to \lambda_0$ and $\|\varphi_\varepsilon - J\varphi_0\|_{\mathscr{H}_\varepsilon} \to 0$.*
- *Convergence of essential spectrum: $\sigma_{\mathrm{ess}}(\Delta_\varepsilon) \to \sigma_{\mathrm{ess}}(\Delta_0)$ converges uniformly in $[0, \Lambda]$ for all $\Lambda > 0$. In particular, Δ_ε has a spectral gap in the essential spectrum if Δ_0 has (provided $\varepsilon > 0$ is small enough).*

O. Post, Spectral analysis on graph-like spaces, Lecture Notes in Mathematics, vol. 2039, Springer, Heidelberg, 2012.

Convergence of Laplacian on thin branched manifolds

We come back to thin branched manifolds and define a suitable identification operator

$$J \colon \mathsf{L}_2(X_0) \longrightarrow \mathsf{L}_2(X_\varepsilon).$$

Assume (for simplicity) that $\mathrm{vol}_m Y_e = 1$ (unscaled transversal volume)

- As identification operator we choose

  $$(Jf)_e = f_e \otimes \mathbb{1}_{\varepsilon,e} \qquad \text{and} \qquad (Jf)_v = 0$$

  where
- $(Jf)_e$ is the contribution on the edge neighbourhoood $X_{\varepsilon,e}$ and
- $(Jf)_v$ is the contribution on the vertex neighbourhood, according to the decomposition (4.1). Moreover, $\mathbb{1}_{\varepsilon,e}$ is the constant function on $Y_{\varepsilon,e}$ with value $\varepsilon^{-m/2}$ (the first normalised eigenfunction of $Y_{\varepsilon,e}$).

Remark

- The setting $(Jf)_v = 0$ seems at first sight a bit rough, but $(Jf)_v = \varepsilon^{-m/2} f(v)$ is meaningless: In $\mathsf{L}_2(X_0)$, the value of f at v is not defined.
- There is a finer version of identification operators on the level of the quadratic form domains, again see [Pos12, Ch. 4] for details.

Resolvent difference

Let us now calculate the resolvent difference $R_\varepsilon J - JR_0$: For $g \in \mathsf{L}_2(X_0)$ and $w \in \mathsf{L}_2(X_\varepsilon)$, we have

$$\begin{aligned}\langle (R_\varepsilon J - JR_0)g, w\rangle_{\mathsf{L}_2(X_\varepsilon)} &= \langle Jg, R_\varepsilon w\rangle_{\mathsf{L}_2(X_\varepsilon)} - \langle JR_0 g, w\rangle_{\mathsf{L}_2(X_\varepsilon)}\\ &= \langle J\Delta_0 f, u\rangle_{\mathsf{L}_2(X_\varepsilon)} - \langle Jf, \Delta_\varepsilon u\rangle_{\mathsf{L}_2(X_\varepsilon)},\end{aligned}$$

where $u = R_\varepsilon w \in \operatorname{dom} \Delta_\varepsilon$ and $f = R_0 g \in \operatorname{dom} \Delta_0$. Moreover, by the definition of Jf,

$$= \frac{1}{2}\sum_{e\in E}\Big(\langle(-f_e'' \otimes \mathbb{1}_{\varepsilon,e}, u_e\rangle_{\mathsf{L}_2(X_{\varepsilon,e})} - \langle f_e \otimes \mathbb{1}_{\varepsilon,e}, -u_e'' + (\mathrm{id}\otimes\Delta_{Y_{\varepsilon,e}})u_e\rangle_{\mathsf{L}_2(X_{\varepsilon,e})}\Big)$$

$$= \frac{1}{2}\sum_{e\in E}\Big(\langle(-f_e'' \otimes \mathbb{1}_{\varepsilon,e}, u_e\rangle_{\mathsf{L}_2(X_{\varepsilon,e})} - \langle f_e \otimes \mathbb{1}_{\varepsilon,e}, -u_e''\rangle_{\mathsf{L}_2(X_{\varepsilon,e})}\Big)$$

since we can bring $(\mathrm{id}\otimes\Delta_{Y_{\varepsilon,e}})$ on the other side of the inner product (the operator is self-adjoint!) and $\Delta_{Y_{\varepsilon,e}}\mathbb{1}_{\varepsilon,e} = 0$. Using $\mathrm{d}X_{\varepsilon,e} = \varepsilon^m \,\mathrm{d}Y_e\,\mathrm{d}s$ and performing a partial integration (Green's first formula), we obtain (u_e' denotes the derivative with respect to the longitudinal variable $s \in M_e$)

$$= \frac{1}{2}\sum_{e\in E}\varepsilon^{m/2}\Big[\int_{Y_e}(-f_e'\overline{u}_e + f_e\overline{u}_e')\,\mathrm{d}Y_e\Big]_{\partial M_e}$$

Resolvent difference II

using our sign convention (2.3) and (2.4), and after reordering,

$$= \frac{1}{2} \sum_{e \in E} \varepsilon^{m/2} \Big[\int_{Y_e} (-f_e' \overline{u}_e + f_e \overline{u}_e') \, \mathrm{d} Y_e \Big]_{\partial M_e}$$

$$= \sum_{v \in V} \sum_{e \in E_v} \varepsilon^{m/2} \int_{Y_e} \big(- \underbrace{f_e'(v) \overline{u}_e(v)}_{=: I_1} + \underbrace{f_e(v) \overline{u}_e'(v)}_{=: I_2} \big) \, \mathrm{d} Y_e.$$

Consider now

$$⨍_v u_v := \frac{1}{\operatorname{vol} X_v} \int_{X_v} u_v \, \mathrm{d} X_v \qquad \text{and} \qquad ⨍_e u_e(v) := \frac{1}{\operatorname{vol} Y_e} \int_{Y_e} u_e(v) \, \mathrm{d} Y_e,$$

then we express the first summand I_1 as

$$\sum_{e \in E_v} \varepsilon^{m/2} \int_{Y_e} f_e'(v) \overline{u}_e(v) = \sum_{e \in E_v} \varepsilon^{m/2} f_e'(v) \big(⨍_e \overline{u}_e(v) - ⨍_v \overline{u}_v \big) + \Big(\sum_{e \in E_v} \varepsilon^{m/2} f_e'(v) \Big) ⨍_v \overline{u}_v$$

$$= \sum_{e \in E_v} \varepsilon^{m/2} f_e'(v) \big(⨍_e \overline{u}_e(v) - ⨍_v \overline{u}_v \big).$$

The last sum in the first line vanishes since $f \in \operatorname{dom} \Delta_0$ fulfils the Kirchhoff condition $\sum_{e \in E_v} f_e'(v) = 0$.

Resolvent difference III

For the second summand I_2, we use the fact that $f_e(v) = f(v)$ is independent of $e \in E_v$ and obtain

$$\sum_{e \in E_v} \varepsilon^{m/2} \int_{Y_e} f_e(v) \overline{u}_e'(v)) \, \mathrm{d} Y_e = \varepsilon^{m/2} f(v) \int_{\partial X_v} \partial_\mathrm{n} \overline{u}_v \, \mathrm{d}\partial X_v = \varepsilon^{m/2} f(v) \int_{X_v} \Delta_{X_v} \overline{u}_v \, \mathrm{d} X_v,$$

performing again a partial integration (Green's first formula, writing u_v as $1 \cdot u_v$). Summing up the contributions, we have

$$\begin{aligned}\langle (R_\varepsilon J - J R_0) g, w \rangle_{\mathsf{L}_2(X_\varepsilon)} &= \sum_{v \in V} \varepsilon^{m/2} \Big(- \sum_{e \in E_v} f_e'(v) \big(⨍_e \overline{u}_e(v) - ⨍_v \overline{u}_v \big) + f(v) \int_{X_v} \Delta_{X_v} \overline{u}_v \, \mathrm{d} X_v \Big) \\ &=: - \langle B_0 g, A_\varepsilon w \rangle_{\mathscr{G}^{\max}} + \langle A_0 g, B_\varepsilon w \rangle_{\mathscr{G}},\end{aligned}$$

where $\mathscr{G} := \ell_2(V, \deg)$, $\mathscr{G}^{\max} := \bigoplus_{v \in V} \mathbb{C}^{E_v}$ and

$$\begin{aligned}
B_0 \colon \mathsf{L}_2(X_0) &\longrightarrow \mathscr{G}^{\max}, & (B_0 g)_v &= \big((R_0 g)_e'(v) \big)_{e \in E_v}, \\
A_\varepsilon \colon \mathsf{L}_2(X_\varepsilon) &\longrightarrow \mathscr{G}^{\max}, & (A_\varepsilon w)_v &= \varepsilon^{m/2} \big(⨍_e (R_\varepsilon w)_e(v) - ⨍_v (R_\varepsilon w)_v \big)_{e \in E_v} \\
B_\varepsilon \colon \mathsf{L}_2(X_\varepsilon) &\longrightarrow \mathscr{G}, & (B_\varepsilon w)(v) &= \frac{\varepsilon^{m/2}}{\deg v} \int_{X_v} \Delta_{X_v} (R_\varepsilon w) \, \mathrm{d} X_v \\
A_0 \colon \mathsf{L}_2(X_0) &\longrightarrow \mathscr{G}, & (A_0 g)(v) &= (R_0 g)(v).
\end{aligned}$$

Resolvent difference IV

In particular, we have shown

Theorem

We can express the resolvent differences of Δ_ε and Δ_0, sandwiched with the identification operator J, as

$$R_\varepsilon J - JR_0 = -A_\varepsilon^* B_0 + B_\varepsilon^* A_0 \colon \mathsf{L}_2(X_0) \longrightarrow \mathsf{L}_2(X_\varepsilon)$$

where $\mathscr{G} := \ell_2(V, \deg)$, $\mathscr{G}^{\max} := \bigoplus_{v \in V} \mathbb{C}^{E_v}$ *and*

$$\begin{aligned}
B_0 &\colon \mathsf{L}_2(X_0) \longrightarrow \mathscr{G}^{\max}, & (B_0 g)_v &= \big((R_0 g)'_e(v)\big)_{e \in E_v},\\
A_\varepsilon &\colon \mathsf{L}_2(X_\varepsilon) \longrightarrow \mathscr{G}^{\max}, & (A_\varepsilon w)_v &= \varepsilon^{m/2}\Big(\textstyle\fint_e (R_\varepsilon w)_e(v) - \fint_v (R_\varepsilon w)_v\Big)_{e \in E_v}\\
B_\varepsilon &\colon \mathsf{L}_2(X_\varepsilon) \longrightarrow \mathscr{G}, & (B_\varepsilon w)(v) &= \frac{\varepsilon^{m/2}}{\deg v} \int_{X_v} \Delta_{X_v}(R_\varepsilon w)\,\mathrm{d}X_v\\
A_0 &\colon \mathsf{L}_2(X_0) \longrightarrow \mathscr{G}, & (A_0 g)(v) &= (R_0 g)(v).
\end{aligned}$$

Two main estimates: Sobolev trace and min-max

Let us now state two important estimates: Sobolev trace estimate We have

$$\|u(0,\cdot)\|^2_{\mathsf{L}_2(Y_e)} \le C(\ell)\|u\|^2_{\mathsf{H}^1(X_{v,e})} \tag{5.2}$$

for all $u \in \mathsf{H}^1(X_{v,e})$, where $X_{v,e} = [0,\ell] \times Y_e$ is a collar neighbourhood of the boundary component of X_v touching the edge neighbourhood X_e. The constant $C(\ell)$ is the same as in the Sobolev trace estimate.
The proof of (5.2) ist just a vector-valued version of (1.4)!
A min-max estimate We have

$$\|u - ⨍ u\|^2_{\mathsf{L}_2(X_v)} \le \frac{1}{\lambda_2(X_v)}\|\mathrm{d}u\|^2_{\mathsf{L}_2(X_v)} \tag{5.3}$$

for all $u \in \mathsf{H}^1(X_v)$, where $\lambda_2(X_v)$ is the first (non-vanishing) Neumann eigenvalue of X_v. Note that $u - ⨍ u$ is the projection onto the space orthogonal to the first (constant) eigenfunction on X_v.

Estimate on the resolvent difference

Proposition *We have (average on boundary $⨍_e$ versus full set $⨍_v$)*

$$\varepsilon^m \sum_{e \in E_v} \Big|⨍_e u_e(v) - ⨍_v u\Big|^2 \le \varepsilon C(\ell_0)\Big(\frac{1}{\lambda_2(X_v)} + 1\Big)\|\mathrm{d}u\|^2_{\mathsf{L}_2(X_{\varepsilon,v})} \quad \textit{for all } u \in \mathsf{H}^1(X_{\varepsilon,v}).$$

Proof.

We have (denoting by $\ell_0 > 0$ a lower bound on the edge lengths)

$$\begin{aligned}
\varepsilon^m \sum_{e \in E_v} \Big|⨍_e u_e(v) - ⨍_v u\Big|^2 &= \varepsilon^m \sum_{e \in E_v} \Big|⨍_e (u - ⨍_v u)\Big|^2 \quad (⨍_e 1 = 1)\\
\text{(Cauchy-Schwarz)} &\le \varepsilon^m \sum_{e \in E_v} \int_{Y_e} \Big|u - ⨍_v u\Big|^2 \mathrm{d}Y_e\\
\text{(Sobolev trace)} &\le \varepsilon^m C(\ell_0) \sum_{e \in E_v} \Big(\|u - ⨍_v u\|^2_{\mathsf{L}_2(X_{v,e})} + \|\nabla u\|^2_{\mathsf{L}_2(X_{v,e})}\Big) \quad (\nabla ⨍_v u = 0)\\
&\le \varepsilon^m C(\ell_0)\Big(\|u - ⨍_v u\|^2_{\mathsf{L}_2(X_v)} + \|\nabla u\|^2_{\mathsf{L}_2(X_v)}\Big) \quad \Big(\bigcup_{e \in E_v} X_{v,e} \subset X_v\Big)\\
\text{(min-max)} &\le \varepsilon C(\ell_0)\Big(\frac{1}{\lambda_2(X_v)} + 1\Big)\|\nabla u\|^2_{\mathsf{L}_2(X_{\varepsilon,v})} \quad (\varepsilon^{m-1}\|\nabla u\|^2_{\mathsf{L}_2(X_v)} = \|\nabla u\|^2_{\mathsf{L}_2(X_{\varepsilon,v})})
\end{aligned}$$

Estimate on the resolvent difference II

The following result is not hard to see using the Sobolev trace lemma (actually, $A_0 = \Gamma R_0$, Γ is evaluation operator in lemma for closedness of form for Δ_M).

Proposition

Assume that $0 < \ell_0 \le \ell_e$ for all $e \in E$, then the operators A_0 and B_0 are bounded by a constant depending only on ℓ_0.

Proposition

Assume that

$$0 < \ell_0 \le \ell_e \; \forall e \in E, \qquad 0 < \lambda_2 \le \lambda_2(X_v) \qquad \textit{and} \qquad \frac{\operatorname{vol} X_v}{\deg v} \le c_{\mathrm{vol}} < \infty \; \forall v \in V,$$

then $\|A_\varepsilon\| = O(\varepsilon^{1/2})$ and $\|B_\varepsilon\| = O(\varepsilon^{3/2})$, and the errors depend only on ℓ_0, λ_0 and c_{vol}.

Estimate on the resolvent difference III

Proof.

For A_ε, we have

$$\|A_\varepsilon w\|^2_{\mathscr{G}^{\max}} = \varepsilon^m \sum_{v \in V} \sum_{e \in E_v} \Big| ⨍_e u_e(v) - ⨍_v u_v \Big|^2$$

$$\leq \varepsilon C(\ell_0)\Big(\frac{1}{\lambda_2} + 1\Big) \sum_{v \in V} \|\nabla u\|^2_{\mathsf{L}_2(X_{\varepsilon,v})} \leq \varepsilon C(\ell_0)\Big(\frac{1}{\lambda_2} + 1\Big) \|\nabla u\|^2_{\mathsf{L}_2(X_\varepsilon)}$$

using the prop. comparing averages, where $u = R_\varepsilon w$. Now, since $u \in \operatorname{dom} \Delta_{X_\varepsilon}$, and since Δ_{X_ε} is the operator associated to the quadratic form, we have

$$\|\nabla u\|^2_{\mathsf{L}_2(X_\varepsilon)} = \langle \Delta_{X_\varepsilon} u, u \rangle_{\mathsf{L}_2(X_\varepsilon)} = \langle \Delta_{X_\varepsilon} (\Delta_{X_\varepsilon} + 1)^{-1} w, (\Delta_{X_\varepsilon} + 1)^{-1} w \rangle_{\mathsf{L}_2(X_\varepsilon)} \leq \|w\|^2_{\mathsf{L}_2(X_\varepsilon)}$$

and the inequality is true by the spectral calculus. □

Estimate on the resolvent difference IV

Proof continued.

For B_ε, we have

$$\begin{aligned}\|B_\varepsilon g\|^2_{\mathscr{G}} = \varepsilon^m \sum_{v\in V} \frac{1}{\deg v}\Big|\int_{X_v} \Delta_{X_v} u\Big|^2 &\le \varepsilon^m \sum_{v\in V} \frac{\operatorname{vol} X_v}{\deg v} \|\Delta_{X_v} u\|^2_{\mathsf{L}_2(X_v)} \quad \text{(Cauchy-Schwarz)}\\ &= \varepsilon^3 \sum_{v\in V} \frac{\operatorname{vol} X_v}{\deg v} \|\Delta_{X_{\varepsilon,v}} u\|^2_{\mathsf{L}_2(X_{\varepsilon,v})}\\ &\le \varepsilon^3 c_{\mathrm{vol}} \|\Delta_{X_\varepsilon}(\Delta_{X_\varepsilon}+1)^{-1} w)\|^2_{\mathsf{L}_2(X_\varepsilon)} \le \varepsilon^3 c_{\mathrm{vol}} \|w\|^2_{\mathsf{L}_2(X_\varepsilon)}\end{aligned}$$

using the scaling behaviour $\Delta_{X_v} = \varepsilon^2 \Delta_{X_{\varepsilon,v}}$ and $\varepsilon^{m+1}\|w\|^2_{\mathsf{L}_2(X_v)} = \|w\|^2_{\mathsf{L}_2(X_{\varepsilon,v})}$, where again $u = R_\varepsilon w$. □

Finale: the main result

Combining the previous results (theorem on resolvent difference and estimates on A_ε and B_ε), we have shown the following:

Theorem (P:06,12,Exner-P:09/13)

Assume that

$$0 < \ell_0 \le \ell_e \; \forall e \in E, \qquad 0 < \lambda_2 \le \lambda_2(X_v) \qquad \text{and} \qquad \frac{\operatorname{vol} X_v}{\deg v} \le c_{\mathrm{vol}} < \infty \; \forall v \in V,$$

then

$$\|R_\varepsilon J - J R_0\|_{\mathsf{L}_2(X_0) \to \mathsf{L}_2(X_\varepsilon)} = \mathrm{O}(\varepsilon^{1/2}),$$

where the error depends only on ℓ_0, λ_0 and c_{vol}.

Theorem

Under the same assumptions as above, the (Neumann) Laplacian Δ_{X_ε} converges to the standard (Kirchhoff) Laplacian Δ_{X_0} in the generalised norm resolvent sense.
In particular, the abstract results apply, i.e., we have convergence of the spectrum (discrete or essential) and we can approximate $\varphi(\Delta_{X_\varepsilon})$ by $J\varphi(\Delta_{X_0})J^*$ in operator norm up to an error of order $\mathrm{O}(\varepsilon^{1/2})$.

Finale: the main result and the missing pieces of its proof

Idea of proof.

We have to show that J is δ_ε-quasi unitary. It is not hard to see that

$$(J^* u)_e(s) = \varepsilon^{m/2} \int_{Y_e} u_e(s, \cdot) \, \mathrm{d} Y_e,$$

and that

$$J^* J f = f$$

for all $f \in \mathsf{L}_2(X_0)$ (i.e., going from the metric graph to the manifold and back, we do not loose information).
Hence we only have to show that

$$\|u - JJ^* u\|^2 = \sum_{v \in V} \|u_v\|^2_{\mathsf{L}_2(X_{\varepsilon,v})} + \sum_{e \in E} \int_{M_e} \|u_e(s,\cdot) - ⨍_e u_e(s,\cdot)\|^2_{\mathsf{L}_2(Y_{\varepsilon,e})} \, \mathrm{d}s \le \delta_\varepsilon^2 \|(\Delta_\varepsilon + 1) u\|^2$$

for some $\delta_\varepsilon \to 0$. Actually, this can be done using similar ideas as before. For details, we refer again to [Pos12, Sec. 6.3], and one can show that $\delta_\varepsilon = \mathrm{O}(\varepsilon^{1/2})$ under the additional assumption that $0 < \lambda_0 \le \lambda_2(Y_e)$ (the first non-zero eigenvalue of Δ_{Y_e} on Y_e). □

Outlook

- One can also treat (magnetic) Schrödinger operators on discrete, metric graphs and thin branched manifolds (see Details can be found in [EP07, Sec. IV, VI] and [EP13])
- Using properly scaled (magnetic) Schrödinger operators on thin branched manifolds, one can basically approximate all type of self-adjoint vertex conditions, see [EP13].

Some open problems

Let us mention here some on-going research of problems, which are still open or at least not completely treated.

- Consider other types of convergences, like convergence of the operators in Hilbert-Schmidt norm: For the wave operators $e^{-t\Delta_{X_\varepsilon}}$, the convergence $e^{-t\Delta_{X_\varepsilon}} J \to J e^{-t\Delta_{X_0}}$ in Hilbert-Schmidt norm is nothing but the L_2-convergence of the heat kernels.
- Consider non-linear operators on metric graphs and thin branched manifolds (there are already some results for this concrete case, see e.g. [Kos00] for a semi-linear equation).
- Consider multi-particle models on metric graphs (there are already some partial results in this case). Can one develop a similar abstract framework for convergence of operators in different spaces in this setting?
- The case of the Dirichlet Laplacian is much more complicated, due to the fact that the lowest transversal Dirichlet eigenvalue is no longer 0, but of order ε^{-2}. In order to obtain a reasonable convergence, on has to rescale the operator $\Delta^{D}_{X_\varepsilon}$ suitably. What is know in this case is an eigenvalue asymptotic (if X_ε is compact), but in most of the cases ("generically"), the limit operator on the metric graph is decoupled (e.g. $\Delta^{D,dec}_{X_0}$, hence physically not very interesting. We refer to [Gri08, MV07, Pos05] for details and [Pos12, Sec. 1.2.2] for a history on this problem and more references.

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