# Diffusion problems on metric graphs

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$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta u(t,x) & t \geq 0,\ x \in \Omega \\ u(0,x) = u_0(x) & x \in \Omega \\ u(t,z) = 0 & t \geq 0,\ z \in \partial\Omega \end{cases}$$

If $\Omega \subset \mathbb{R}^d$ is open, Lipschitz, bounded, then $\Delta$ with Dirichlet BCs is self-adjoint and negative semidefinite, and it has compact resolvent:

- the eigenvalues $\lambda_k$, $k \in \mathbb{N}$, of $-\Delta$ have finite multiplicities and accumulate at $+\infty$
- there exists an ONB of $L^2(\Omega)$ consisting of corresponding eigenfunctions $\varphi_k$, $k \in \mathbb{N}$.

- Spectral Theorem:

$$\begin{aligned} u(t,x) &= \mathrm{e}^{t\Delta} u_0(x) \\ &= \sum_{k\in\mathbb{N}} \mathrm{e}^{-t\lambda_k} \langle \varphi_k, u_0 \rangle_{L^2(\Omega)} \varphi_k(x) \\ &= \int_\Omega \sum_{k\in\mathbb{N}} \mathrm{e}^{-t\lambda_k} \varphi_k(x)\varphi_k(y) u_0(y)\,\mathrm{d}y \\ &=: \int_\Omega p_t(x,y) u_0(y)\,\mathrm{d}y \end{aligned}$$

- $\mathrm{e}^{t\Delta}$ is compact, self-adjoint, and positive definite
  $\rightsquigarrow$ Mercer's Theorem: the series

$$p_t(x,y) := \sum_{k\in\mathbb{N}} \mathrm{e}^{-t\lambda_k} \varphi_k(x)\varphi_k(y)$$

  converges absolutely and uniformly in $\overline{\Omega} \times \overline{\Omega}$, for all $t > 0$.

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# Heat kernels

$(X, d, \mu)$ metric measure space, $A$ operator on $L^p(X;\mu)$

$p = p_t(x,y) : (0,\infty) \times X \times X \to \mathbb{C}$ is the **heat kernel** associated with $A$ if $\forall t > 0$, $\forall x, y \in X$

1. $p_t(x,\cdot)f(\cdot) \in L^1(X)$ for all $f \in L^p(X)$
2. $t \mapsto p_t(\cdot,y) \in C^1\left((0,\infty); L^p(X)\right) \cap C\left((0,\infty); D(A_x)\right)$
3. $\frac{\partial}{\partial t} p_t(\cdot,y) = A_x p_t(\cdot,y)$
4. $p_{t+s}(x,y) = \int_X p_t(x,z)p_s(z,y)\,\mathrm{d}\mu(z)$
5. $\lim\limits_{t\to 0^+} \int_X p_t(\cdot,y)f(y)\,\mathrm{d}\mu(y) = f(\cdot)$ (in $L^p(X)$) for all $f \in L^p(X)$

Let $A$ be differential operator on $L^2(\Omega)$ (with BC)

- If there is a heat kernel associated with $A$, then

$$(*) \qquad \begin{cases} \dfrac{\partial u}{\partial t}(t,x) = Au(t,x) & t \geq 0,\ x \in \Omega \\ u(0,x) = u_0(x) & x \in \Omega \end{cases}$$

  is well-posed.

- $(*)$ well-posed $\not\Rightarrow$ $A$ has a heat kernel: e.g. $\Omega = \mathbb{R}$, $A = \frac{\partial}{\partial x}$.
  Then $u(t,x) = \int_{\mathbb{R}} \delta_{x+t}(y) u_0(y)\, \mathrm{d}y$
  but $p_t(\cdot, y) = \delta_{\cdot+t}(y) \notin H^1(\mathbb{R})$

- $A$ has a heat kernel $\not\Rightarrow$

$$p_t(x,y) = \sum_{k \in \mathbb{N}} \mathrm{e}^{-t\lambda_k} \varphi_k(x) \varphi_k(y)$$

  e.g. $\Omega = \mathbb{R}$, $A = \frac{\partial^2}{\partial x^2}$, $p_t(x,y) = \frac{1}{\sqrt{4\pi t}} \mathrm{e}^{-\frac{|x-y|^2}{4t}}$ but no eigenvalues

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Even if

$$p_t(x,y) = \sum_{k\in\mathbb{N}} \mathrm{e}^{-t\lambda_k}\varphi_k(x)\varphi_k(y),$$

this may be difficult to use to deduce information on the heat equation.

However,

- $p_t(\cdot,\cdot) > 0 \ \forall t \Leftrightarrow$ parabolic strict maximum principle
  (i.e., $u_0 \geq 0,\ u \not\equiv 0 \Rightarrow u(t,\cdot) > 0 \ \forall t$)
- $0 \leq p_t(\cdot,\cdot) \leq 1 \ \forall t \Leftrightarrow$ Markov property
  (i.e., $0 \leq u_0 \leq 1 \Rightarrow 0 \leq u(t,\cdot) \leq 1 \ \forall t$)
- $|p_t^{(1)}(x,y)| \leq p_t^{(2)}(x,y) \Leftrightarrow$ domination
  (i.e., $|u_0^{(1)}| \leq u_0^{(2)} \Rightarrow |u^{(1)}(t)| \leq u^{(2)}(t) \ \forall t$)
- $p_t(\cdot,\cdot) \in C^\infty(X\times X) \ \forall t>0 \Leftrightarrow$ smoothing effect
  (i.e., $u_0 \in \mathcal{D}'(X) \Rightarrow u(t,\cdot) \in C^\infty(X)$); Schwartz–Hörmander

### Theorem

*Given $\mathcal{G}$ on finitely many edges of finite length, the Laplacian $\Delta_{\mathcal{G}}$ on $\mathcal{G}$ generates an analytic $C_0$-semigroup on $L^2(\mathcal{G})$. Indeed, it is associated with a heat kernel $p^{\mathcal{G}} = p_t^{\mathcal{G}}(x,y)$ that satisfies.*

- $0 \le p_t^{\mathcal{G}}(x,y) \le 1$ *for all* $t$ *and all* $x, y \in \mathcal{G}$;
- *if* $\mathcal{G}$ *is connected,* $0 < p_t(x,y)$ *for all* $t$ *and all* $x, y \in \mathcal{G}$;
- *if Dirichlet conditions are imposed on a subset* $\mathrm{V}^{\mathrm{D}} \subset \mathrm{V}$, $p_t^{\mathcal{G};\mathrm{V}^{\mathrm{D}}}(x,y) \le p_t^{\mathcal{G}}(x,y)$;
- *both* $p_t^{\mathcal{G}}$ *and* $\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}p_t^{\mathcal{G}}$ *are jointly Lipschitz continuous, but* $p_t^{\mathcal{G}}(\cdot, y)$ *is not continuously differentiable for any* $y$ *unless* $\mathcal{G}$ *is a loop or a path.*

# $C_0$-semigroups

## Definition

*Let $E$ be a normed space. A $C_0$-**semigroup** is a family $(T(t))_{t\geq 0}$ of bounded linear operators on $E$ such that*

- $T(0) = \mathrm{Id}$
- $T(t+s) = T(t)T(s)$
- $\lim_{t\to 0} T(t)f = f$ *for all* $f \in E$.

## Example

$T(t)f(\cdot) = f(t+\cdot)$ is a $C_0$ semigroup on $E = L^p(\mathbb{R})$ for any $p \in [1,\infty)$ (but not for $p=\infty$: Exercise).

## Example

$T(t)f(\cdot) = \mathrm{e}^{tq(\cdot)}f(\cdot)$ is a $C_0$ semigroup on $E = L^p(\Omega)$ for any $p \in [1,\infty)$ and any $q \in L^\infty(X)$.

# Generators

## Definition

*An operator $A$ on $E$ is said to be a* **generator** *of a $C_0$-semigroup $(T(t))_{t\geq 0}$ on $E$ if*

$$D(A) = \left\{ f \in E : \exists \lim_{t \geq 0+} \frac{T(t)f - f}{t} \right\}$$

$$Af = \lim_{t \geq 0+} \frac{T(t)f - f}{t}.$$

## Example

$T(t)f(\cdot) = f(t + \cdot)$ on $L^p(\mathbb{R})$ is generated by

$$D(A) = W^{1,p}(\mathbb{R})$$

$$Af = f'.$$

## Example

$T(t)f(\cdot) = \mathrm{e}^{tq(\cdot)} f(\cdot)$ on $L^p(\Omega)$, $\Omega \subset \mathbb{R}^d$ is generated by

$$D(A) = L^p(\Omega)$$

$$Af = qf.$$

Recall:
$p = p_t(x,y) : (0,\infty) \times X \times X \to \mathbb{C}$ is the **heat kernel** associated with $A$ if $\forall t > 0$, $\forall x, y \in X$

1. $p_t(x,\cdot)f(\cdot) \in L^1(X)$ for all $f \in L^p(X)$
2. $t \mapsto p_t(\cdot,y) \in C^1\left((0,\infty); L^p(X)\right) \cap C\left((0,\infty); D(A_x)\right)$
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## Example

If there is a heat kernel $p$ associated with $A$, then $A$ generates on $E = L^2(X;\mu)$ a $C_0$-semigroup given by

$$T(t)f = \int_X p_t(\cdot,y)f(y)\,\mathrm{d}\mu(y), \qquad t \geq 0.$$

### Proposition

*For a generator $A$ of a $C_0$-semigroup $(T(t))_{t\geq 0}$ on $E$ the following hold:*

- *$A$ is linear;*
- *if $f \in D(A)$, then $T(t)f \in D(A)$ and $\frac{\mathrm{d}}{\mathrm{d}t}T(t)f = T(t)Af = AT(t)f$ for all $t \geq 0$;*
- *$A$ is closed and densely defined;*
- *$(T(t))_{t\geq 0}$ determines its generator uniquely, and vice versa.*

### Proof.

Exercise □

The $C_0$-semigroup generated by $A$ is denoted by $(\mathrm{e}^{tA})_{t\geq 0}$.

# Analytic semigroups

**Definition**

*A $C_0$-semigroup $(\mathrm{e}^{tA})_{t\geq 0}$ on a Banach space $E$ is called* **analytic** *if*

$$\|tA\mathrm{e}^{tA}f\| \leq c\|f\|$$

*for some $c > 0$ and all $t \in (0,1]$ and $f \in D(A)$.*

In particular,

$$\|A\mathrm{e}^{tA}f\| \leq c(t)\|f\|$$

i.e., $\mathrm{e}^{tA}$ is bounded from $E$ to $D(A)$, hence (Exercise) from $E$ to $\bigcap_{k\in\mathbb{N}} D(A^k)$, for all $t > 0$.

## Example

- $T(t)f(\cdot) = \mathrm{e}^{tq(\cdot)f(\cdot)}$ is analytic, for any $q \in L^\infty(\Omega)$;
- $T(t)f(\cdot) = f(t+\cdot)$ is NOT analytic.

## Remark

A $C_0$-semigroup $(\mathrm{e}^{t\Delta^{\mathcal{G}}})_{t\geq 0}$ is analytic if and only if for some $\theta \in (0,\pi)$ it has an analytic extension $(\mathrm{e}^{t\Delta^{\mathcal{G}}})_{t\in\Sigma_\theta}$ that is bounded on $\Sigma_\theta \cap \{z \in \mathbb{C} : |z| \leq 1\}$, where

$$\Sigma_\theta := \{r\mathrm{e}^{i\alpha} : r > 0,\ |\alpha| < \theta\}.$$

Any closed quadratic form $\mathfrak{A}$ on $L^2(X)$ is associated with a unique self-adjoint, positive semi-definite operator $A$ on $L^2(X)$, and vice versa: there holds

$$\begin{aligned} D(A) &= \{f \in D(\mathfrak{A}) : \exists g \in L^2(X) \text{s.t. } \mathfrak{a}(f,h) = (g,h)\ \forall h \in D(\mathfrak{a})\} \\ Af &= -g \end{aligned}$$

where $\mathfrak{a}$ is the bilinear form corresponding with $\mathfrak{A}$, i.e., $\mathfrak{A}(f) = \frac{1}{2}\mathfrak{a}(f,f)$.
Furthermore, $A$ has compact resolvent iff $D(\mathfrak{A})$ is compactly embedded in $L^2(X;\mu)$.

# Self-adjoint operators and the Spectral Theorem

Let $A$ be a self-adjoint, negative semidefinite operator on $L^2(X;\mu)$ with compact resolvent.
Then

- $L^2(X;\mu)$ has an ONB of eigenvectors of $A$: $(-\lambda_k, \varphi_k)_{k\in\mathbb{N}}$;
- $A$ can be diagonalized:

$$D(A) = \left\{ f \in L^2(X;\mu) : \sum_{k\in\mathbb{N}} \lambda_k^2 (f,\varphi_k)^2 < \infty \right\},$$
$$Af = -\sum_{k\in\mathbb{N}} \lambda_k (f,\varphi_k)\varphi_k$$

- $A$ is associated with a closed quadratic form $\mathfrak{A}$ given by

$$D(\mathfrak{a}) = \left\{ f \in L^2(X;\mu) : \sum_{k\in\mathbb{N}} \lambda_k (f,\varphi_k)^2 < \infty \right\}$$
$$\mathfrak{a}(f,g) = \sum_{k\in\mathbb{N}} \lambda_k (f,\varphi_k)(\varphi_k, g).$$

**!** $\lambda_k \geq 0$

# Semigroups associatd with closed quadratic forms

## Proposition

*Every self-adjoint, negative semidefinite operator generates an analytic semigroup.*

## Proof.

For simplicity, only for operators with compact resolvent:

- By functional calculus, $\mathrm{e}^{tA} := \sum_{k\in\mathbb{N}} \mathrm{e}^{-t\lambda_k}(f,\varphi_k)\varphi_k$ is a well-defined bounded linear operator on $L^2(X;\mu)$;
- Given $f \in D(A)$ and $t > 0$

$$\|tA\mathrm{e}^{tA}f\|^2 = \|t\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tA}f\|^2 = \sum_{k\in\mathbb{N}} |t\lambda_k \mathrm{e}^{-t\lambda_k}(f,\varphi_k)|^2 \leq \frac{1}{\mathrm{e}}\|f\|^2$$

□

# Introducing metric graphs

Figure: Valentina Vetturi, *Tails*, 2023

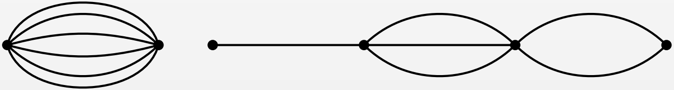
# Introducing metric graphs

Let

- $\mathsf{E} = \{\mathsf{e}_1, \mathsf{e}_2, \ldots\}$ finite or countably infinite set ("edge set")
- $\ell : \mathsf{E} \to (0, \infty)$ ("edge lengths")
- $\sim$ equivalence relation on $\mathcal{V} := \bigsqcup_{\mathsf{e}\in\mathsf{E}} \{0, \ell_\mathsf{e}\}$ ("wiring")

  Define $\mathcal{E} := \bigsqcup_{\mathsf{e}\in\mathsf{E}} [0, \ell_\mathsf{e}]$ and extend canonically $\sim$ to $\mathcal{E}$.

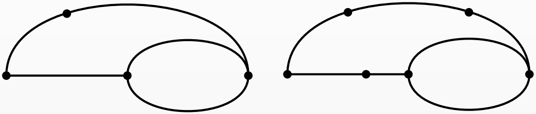
  Then $\mathcal{G} := {}^{\mathcal{E}}\!/\!_{\sim}$ is a **metric graph** and $\mathsf{V} := {}^{\mathcal{V}}\!/\!_{\sim}$ its **vertex set**.



$\mathsf{G} := (\mathsf{V}, \mathsf{E})$ is the **underlying combinatorial graph** of $\mathcal{G}$.

All topological features (number $\kappa$ of connected components, Betti number $\beta := \#\mathsf{E} - \#\mathsf{V} + \kappa$, etc.) are determined by $\sim$.

The metric measure structure of $\mathcal{G}$ does not change upon insertion of artificial, degree-2 vertices.



Inserting degree-2 vertices defines an equivalence relation. We will not distinguish between a metric graph and any of its representatives.

A metric graph does not have an intrinsic notion of boundary[1], but each of its subgraphs does.



[1]Not even the vertices of degree 1 are consistently "boundary"! E.g., the hot spot conjecture dramatically fails for metric graphs: the hot spots need not be located at vertices of degree 1.

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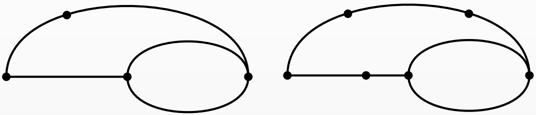

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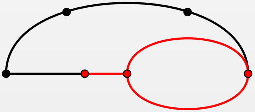

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Goal: define a Laplacian on $\mathcal{G}$ by means of a quadratic function on $L^2(\mathcal{G})$.
Idea: integrate $-\Delta^{\mathcal{G}} f \in L^2(\mathcal{G})$ against a test function $h \in C(\mathcal{G}) \cap L^2(\mathcal{G})$.

$$
\begin{aligned}
(-\Delta^{\mathcal{G}} f, h) &= \int_{\mathcal{G}} f''(x) h(x)\,\mathrm{d}x \\
&= -\sum_{\mathsf{e}\in\mathsf{E}} \int_0^{\ell_\mathsf{e}} f_\mathsf{e}''(x) h_\mathsf{e}(x)\,\mathrm{d}x \\
&= -\sum_{\mathsf{e}\in\mathsf{E}} f_\mathsf{e}'(x) h_\mathsf{e}(x)\,\mathrm{d}x \Big|_{x=0}^{x=\ell_\mathsf{e}} + \sum_{\mathsf{e}\in\mathsf{E}} \int_0^{\ell_\mathsf{e}} f_\mathsf{e}'(x) h_\mathsf{e}'(x)\,\mathrm{d}x \\
&\overset{!}{=} -h(\mathsf{v}) \sum_{\mathsf{e}\sim\mathsf{v}} \frac{\partial f_\mathsf{e}}{\partial n}(\mathsf{v}) + \sum_{\mathsf{e}\in\mathsf{E}} \int_0^{\ell_\mathsf{e}} f_\mathsf{e}'(x) h_\mathsf{e}'(x)\,\mathrm{d}x \\
&\overset{?}{=} \sum_{\mathsf{e}\in\mathsf{E}} \int_0^{\ell_\mathsf{e}} f_\mathsf{e}'(x) h_\mathsf{e}'(x)\,\mathrm{d}x = a(f, h)
\end{aligned}
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&\overset{!}{=} -h(v) \sum_{e \sim v} \frac{\partial f_e}{\partial n}(v) + \sum_{e \in E} \int_0^{\ell_e} f_e'(x) h_e'(x)\,\mathrm{d}x \\
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$$

Consider

$$H^1(\mathcal{G}) := \{f \in C(\mathcal{G}) \cap L^2(\mathcal{G}) : f' \in L^2(\mathcal{G})\}$$

and

$$D(\Delta^{\mathcal{G}}) := -\left\{ f \in H^1(\mathcal{G}) \cap \bigoplus_{e \in E} H^2(0, \ell_e) : \sum_{e \sim v} \frac{\partial f_e}{\partial n}(v) = 0 \ \forall v \in V \right\}$$

### Proposition (Pavlov–Faddeev 1983, Nicaise 1986)

*$\Delta^{\mathcal{G}}$ is a self-adjoint operator on $L^2(\mathcal{G})$ with compact resolvent.*

### Proof.

- It suffices to prove that $\Delta^{\mathcal{G}}$ is associated with the closed quadratic form $\mathfrak{a}^{\mathcal{G}}(f, g) := \int_{\mathcal{G}} f'(x) g'(x) \, \mathrm{d}x$ with domain $D(\mathfrak{a}^{\mathcal{G}}) := H^1(\mathcal{G})$.
- Already proved: $\Delta^{\mathcal{G}} \subset A$. Exercise: prove $A \subset \Delta^{\mathcal{G}}$.
- $D(\mathfrak{a}^{\mathcal{G}}) = H^1(\mathcal{G}) \subset \bigoplus_{e \in E} H^1(0, \ell_e) \overset{c}{\hookrightarrow} \bigoplus_{e \in E} L^2(0, \ell_e) = L^2(\mathcal{G})$.

□

### Remark

More generally, every bounded elliptic bilinear form $\mathfrak{a}$ on $L^2(X; \mu)$ is associated with an operator that generates an analytic semigroup on $L^2(X; \mu)$; the generator is self-adjoint iff $\mathfrak{a}$ is symmetric.

# Useful information about heat kernel on metric graphs?

Hardly so. Explicit construction of the heat kernel of $(\mathrm{e}^{t\Delta^{\mathcal{G}}})_{t\geq 0}$ actually available, via parametrix; however, the formula yields a hardly tractable series.

**Proposition (Roth 1984; Becker–Gregorio–M. 2021)**

*$\Delta^{\mathcal{G}}$ is associated with a heat kernel $p^{\mathcal{G}}$ given by*

$$p_t^{\mathcal{G}}(x,y) := \frac{1}{\sqrt{4\pi t}} \sum_{\gamma\in\mathcal{P}_{x,y}} \alpha(\gamma)\mathrm{e}^{-\frac{\mathrm{length}(\gamma)^2}{4t}}$$

*for appropriate "scattering coefficients" $\alpha(P) \in [-1,1]$.*

Also already known:

$$p_t^{\mathcal{G}}(x,y) := \sum_{k\in\mathbb{N}} \mathrm{e}^{-t\lambda_k}\varphi_k^{\mathcal{G}}(x)\varphi_k^{\mathcal{G}}(y)$$

(uniformly in $\mathcal{G}\times\mathcal{G}$, for all $t>0$).

# Markovian property

## Proposition (Kramar–M.–Sikolya 2007)

$(e^{t\Delta^{\mathcal{G}}})_{t\geq 0}$ *is a Markovian semigroup; it satisfies a strict maximum principle if* $\mathcal{G}$ *is connected.*

## Proof.

- Beurling–Deny 1959: If $A \sim \mathfrak{a}$, and $\mathfrak{a} \geq 0$, then $(e^{tA})_{t\geq 0}$ is Markovian iff $f \in D(\mathfrak{a})$ implies $f \wedge \mathbf{1} \in D(\mathfrak{a})$ and $\mathfrak{a}(f \wedge \mathbf{1}, (f-\mathbf{1})^+) \geq 0$.
- Ouhabaz 1996: If $A \sim \mathfrak{a}$, and if $(e^{tA})_{t\geq 0}$ is positive, then $(e^{tA})_{t\geq 0}$ satisfies the strict maximum principle iff for each measurable $\omega \subset X$ $\mu(\omega) = 0$ or $\mu(X \setminus \omega) = 0$ whenever $\mathbf{1}_\omega f \in D(\mathfrak{a})$ for every $f \in D(\mathfrak{a})$.
- $f_e \in H^1(0, \ell_e)$ implies $f_e \wedge \mathbf{1} \in H^1(0, \ell_e)$ and

$$\int_0^{\ell_e} (f_e \wedge \mathbf{1})'(x)(f_e - \mathbf{1})^+)'(x)\,dx = \int_{\{f\leq 1\}} (f_e \wedge \mathbf{1})'(x)(f_e - \mathbf{1})^+)'(x)\,dx = 0.$$

- Also, $\mathbf{1}_{\omega_e} f \notin H^1(0, \ell_e) \hookrightarrow C[0, \ell_e]$ unless $\omega_e = \emptyset$ or $\omega_e(0, \ell_e)$.
- To conclude, observe that $f \in C(\mathcal{G})$ implies $f \wedge \mathbf{1} \in C(\mathcal{G})$.

□

# Domination

A $C_0$-semigroup $(T(t))_{t\geq 0}$ on $L^p(X)$ is said to **dominate** another $C_0$-semigroup $(S(t))_{t\geq 0}$ if $|S(t)f| \leq T(t)|f|$ for all $f \in L^p(X)$ and all $t \geq 0$.

## Proposition

*Upon imposing Dirichlet conditions on* $\mathrm{V}^{\mathrm{D}} \subset \mathrm{V}$ *we obtain a new* $C_0$*-semigroup* $(\mathrm{e}^{t\Delta^{\mathcal{G};\mathrm{V}^{\mathrm{D}}}})_{t\geq 0}$ *that is dominated by* $(\mathrm{e}^{t\Delta^{\mathcal{G}}})_{t\geq 0}$.

## Exercise (Diamagnetic inequality for point interactions)

*Same holds if magnetic vertex conditions*

$$u(\mathsf{v}+) = \mathrm{e}^{i\theta_{\mathsf{v}}} u(\mathsf{v}-)$$

*are imposed on finitely many vertices* $\mathrm{V}^{\mathrm{m}}$ *of degree 2.*

Given two subspaces $U, V$ of $L^2(X;\mu)$, $U$ is a **generalized ideal** of $V$ if

- $u \in U \Rightarrow |u| \in V$
- $u \in U$, $v \in V$, $|v| \leq |u| \Rightarrow v \,\mathrm{sgn}\, u \in U$.

## Example

$H^1_{antiper}(0,1)$ is a generalized ideal of $H^1_{per}(0,1)$; neither of them is a generalized ideal of $H^1(0,1)$, but $H^1_0(0,1)$ is.

# Proof

- Ouhabaz 1996 : Let $A \sim \mathfrak{a}$ , $B \sim \mathfrak{b}$, $S \sim s$. If $\mathfrak{a}, \mathfrak{b}$ are both restrictions of $\mathfrak{s}$, and if $(\mathrm{e}^{tA})_{t\geq 0}, (\mathrm{e}^{tS})_{t\geq 0}$ are both positive, then $(\mathrm{e}^{tA})_{t\geq 0}$ dominates $(\mathrm{e}^{tB})_{t\geq 0}$ iff $D(\mathfrak{b})$ is a generalized ideal of $D(\mathfrak{a})$.
- If Dirichlet conditions are imposed on $\mathsf{V}^{\mathrm{D}} \subset \mathsf{V}$, then the corresponding operator $\Delta^{\mathcal{G};\mathsf{V}^{\mathrm{D}}}$ is associated with the quadratic form

$$\mathfrak{b}(f,g) = \mathfrak{a}(f,g), \qquad f,g \in D(\mathfrak{b}) := H^1_0(\mathcal{G};\mathsf{V}^{\mathrm{D}})$$

where $H^1_0(\mathcal{G};\mathsf{V}^{\mathrm{D}}) := \{f \in H^1(\mathcal{G}) : f(\mathsf{v}) = 0\ \forall \mathsf{v} \in \mathsf{V}^{\mathrm{D}}\}$.
- Let us check Ouhabaz' criterion: introduce

$$\mathfrak{s}(f,g) = \int_{\mathcal{G}} f'(x)g'(x)\,\mathrm{d}x, \qquad f,g \in D(\mathfrak{s}) := \bigoplus_{\mathsf{e}\in\mathsf{E}} H^1(0,\ell_{\mathsf{e}})$$

which satisfies the Beurling–Deny criterion.
- $H^1_0(\mathcal{G};\mathsf{V}^{\mathrm{D}})$ is a generalized ideal of $H^1(\mathcal{G})$: $f \in H^1_0(\mathcal{G};\mathsf{V}^{\mathrm{D}}) \Rightarrow |f| \in H^1(\mathcal{G})$; and $|g| \leq |f|$ with $f \in H^1_0(\mathcal{G};\mathsf{V}^{\mathrm{D}}) \Rightarrow g\,\mathrm{sgn}\, f \in H^1_0(\mathcal{G};\mathsf{V}^{\mathrm{D}})$.

**Theorem (Kramar–M.–Sikolya 2007, M.–Romanelli 2007, Bifulco–M. 2023)**

*Given $\mathcal{G}$ on finitely many edges of finite length, the Laplacian $\Delta_{\mathcal{G}}$ on $\mathcal{G}$ is associated with a heat kernel $p^{\mathcal{G}} = p_t^{\mathcal{G}}(x, y)$ that satisfies.*

- $0 \leq p_t^{\mathcal{G}}(x, y) \leq 1$ *for all* $t$ *and all* $x, y \in \mathcal{G}$*;*
- *if* $\mathcal{G}$ *is connected,* $0 < p_t(x, y)$ *for all* $t$ *and all* $x, y \in \mathcal{G}$*;*
- *if Dirichlet conditions are imposed on a subset* $\mathrm{V}^{\mathrm{D}} \subset \mathcal{G}$*,* $p_t^{\mathcal{G};\mathrm{V}^{\mathrm{D}}}(x, y) \leq p_t^{\mathcal{G}}(x, y)$*;*
- *both* $p_t^{\mathcal{G}}$ *and* $\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2} p_t^{\mathcal{G}}$ *are jointly Lipschitz continuous, but* $p_t^{\mathcal{G}}(\cdot, y)$ *is not continuously differentiable for any* $y$ *unless* $\mathcal{G}$ *is a loop or a path.*

# Smoothness of functions in $D(\Delta^{\mathcal{G}})$

## Lemma (M.–Plümer 2023)

*$D(\Delta^{\mathcal{G}})$ is continuously embedded in $\mathrm{Lip}(\mathcal{G})$.*

## Proof.

- $D(\Delta^{\mathcal{G}}) \hookrightarrow C(\mathcal{G}) \cap \bigoplus_{e \in E} H^2(0, \ell_e) \hookrightarrow C(\mathcal{G}) \cap \bigoplus_{e \in E} W^{1,\infty}(0, \ell_e)$.
- Let $u \in C(\mathcal{G}) \cap \bigoplus_{e \in E} W^{1,\infty}(0, \ell_e)$. Let $x, y \in \mathcal{G}$ and let $\gamma \subset \mathcal{G}$ be a path connecting $x$ and $y$. Then
$$|u(x) - u(y)| = \left| \int_\gamma u'(t)\,\mathrm{d}t \right| \le \mathrm{length}(\gamma) \|u'\|_\infty .$$
- $\gamma$ arbitrary $\Rightarrow$
$$|u(x) - u(y)| \le \|u'\|_\infty d^{\mathcal{G}}(x, y).$$

Therefore, $C(\mathcal{G}) \cap \bigoplus_{e \in E} W^{1,\infty}(0, \ell_e) \hookrightarrow \mathrm{Lip}(\mathcal{G})$.

□

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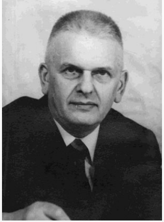
- *if* $\mathcal{G}$ *is connected,* $0 < p_t(x,y)$ *for all* $t$ *and all* $x, y \in \mathcal{G}$;
- *if Dirichlet conditions are imposed on a subset* $\mathrm{V}^{\mathrm{D}} \subset \mathcal{G}$, $p_t^{\mathcal{G};\mathrm{V}^{\mathrm{D}}}(x,y) \leq p_t^{\mathcal{G}}(x,y)$;
- *both* $p_t^{\mathcal{G}}$ *and* $\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2} p_t^{\mathcal{G}}$ *are jointly Lipschitz continuous, but* $p_t^{\mathcal{G}}(\cdot, y)$ *is not continuously differentiable for any* $y$ *unless* $\mathcal{G}$ *is a loop or a path.*

# Proof - #1

- Kantorovič–Wulich: Given $p \in [1, \infty)$, any operator in $\mathcal{L}(L^p(X); L^\infty(X))$ has an integral kernel of class $L^\infty(X; L^{p'}(X))$, and vice versa.

Leonid Vital'evič Kantorovič
1912–1986



Boris Sacharowitsch Wulich
1913–1978

- $D(\Delta^{\mathcal{G}}) \hookrightarrow \mathrm{Lip}(\mathcal{G}) \hookrightarrow L^\infty(\mathcal{G})$: Therefore, $\mathrm{e}^{t\Delta^{\mathcal{G}}}(L^2(\mathcal{G})) \subset L^\infty(\mathcal{G})$ and by duality $\mathrm{e}^{t\Delta^{\mathcal{G}}}(L^1(\mathcal{G})) \subset L^2(\mathcal{G})$: by the semigroup law $\mathrm{e}^{t\Delta^{\mathcal{G}}}(L^1(\mathcal{G})) \subset L^\infty(\mathcal{G})$, i.e., $\mathrm{e}^{t\Delta^{\mathcal{G}}}$ has a heat kernel $p_t^{\mathcal{G}} \in L^\infty(\mathcal{G} \times \mathcal{G})$, for all $t > 0$.
- $p_t(\cdot, y) \in D(\Delta^{\mathcal{G}})$ for all $t > 0$, but functions in $D(\Delta^{\mathcal{G}})$ are not differentiable at any vertex of degree $\geq 3$.

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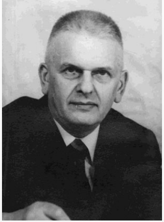

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- $p_t(\cdot, y) \in D(\Delta^{\mathcal{G}})$ for all $t > 0$, but functions in $D(\Delta^{\mathcal{G}})$ are not differentiable at any vertex of degree $\geq 3$.

# Proof - #2

- Let $t > 0$ and $f \in L^2(\mathcal{G})$. Because (i) $D(\Delta^{\mathcal{G}}) \hookrightarrow \mathrm{Lip}(\mathcal{G})$ and (ii) $\mathrm{e}^{t\Delta^{\mathcal{G}}}$ is bounded from $L^2(X;\mu)$ to $D(\Delta^{\mathcal{G}})$

$$
\begin{aligned}
|\mathrm{e}^{t\Delta^{\mathcal{G}}} f(x) - \mathrm{e}^{t\Delta^{\mathcal{G}}} f(x')| &\leq C(t) d^{\mathcal{G}}(x,x') \|\mathrm{e}^{t\Delta^{\mathcal{G}}} f\|_{D(\Delta^{\mathcal{G}})} \\
&\leq \frac{C(t)}{t\mathrm{e}} d^{\mathcal{G}}(x,x') \|f\|_{L^2(\mathcal{G})} \quad \forall x, x' \in \mathcal{G}.
\end{aligned}
$$

- Hence, for all $f \in L^2(\mathcal{G})$

$$
\begin{aligned}
\Big|\big(f, (p_t(x,\cdot) - p_t(x',\cdot))\big)\Big| &= \left| \int_{\mathcal{G}} f(y) \big(p_t(x,y) - p_t(x',y)\big) \,\mathrm{d}y \right| \\
&= \left| \int_{X} f(y) \big(p_t(x,y) - p_t(x',y)\big) \,\mathrm{d}y \right| \\
&= \left| \mathrm{e}^{t\Delta^{\mathcal{G}}} \big(f(x) - f(x')\big) \right| \\
&\leq C'(t) d^{\mathcal{G}}(x,x') \|f\|_{L^2(\mathcal{G})}.
\end{aligned}
$$

- We finally conclude that

$$
\begin{aligned}
\big\|p_t(x,\cdot) - p_t(x',\cdot)\big\|_{L^2(X;\mu)} &= \sup_{\|f\|_{L^2}=1} \Big|\big(f, (p_t(x,\cdot) - p_t(x',\cdot))\big)\Big| \\
&\leq C'(t) d^{\mathcal{G}}(x,x').
\end{aligned}
$$

# Proof - #3

- By the semigroup law

$$p_t(x,y) = \int_{\mathcal{G}} p_{\frac{t}{2}}(x,z) p_{\frac{t}{2}}(z,y)\,\mathrm{d}z$$

whence for a.e. $y \in \mathcal{G}$

$$\begin{aligned} |p_t(x,y) - p_t(x',y)| &\leq C'(\frac{t}{2}) \|p_{\frac{t}{2}}(\cdot,y)\|_{L^2(\mathcal{G})} d^{\mathcal{G}}(x,x') \\ &\leq C''(\frac{t}{2}) \|p_{\frac{t}{2}}\|_{L^\infty(\mathcal{G}\times\mathcal{G})} d^{\mathcal{G}}(x,x'), \end{aligned}$$

i.e., $\mathcal{G} \ni x \mapsto p_t(x,\cdot) \in L^\infty(\mathcal{G})$ is Lipschitz.

- Finally,

$$\begin{aligned} \left|p_t(x,y) - p_t(x',y')\right| &= \left| \int_X p_{\frac{t}{2}}(x,z) p_{\frac{t}{2}}(z,y)\,\mathrm{d}z - \int_X p_{\frac{t}{2}}(x',z) p_{\frac{t}{2}}(z,y')\,\mathrm{d}z \right| \\ &\leq \left\|p_{\frac{t}{2}}(x,\cdot)\right\|_{L^2(\mathcal{G})} \left\|p_{\frac{t}{2}}(\cdot,y) - p_{\frac{t}{2}}(\cdot,y')\right\|_{L^2(\mathcal{G})} \\ &\quad + \left\|p_{\frac{t}{2}}(\cdot,y')\right\|_{L^2(\mathcal{G})} \left\|p_{\frac{t}{2}}(x,\cdot) - p_{\frac{t}{2}}(x',\cdot)\right\|_{L^2(\mathcal{G})} \\ &\leq C'''(\frac{t}{2})(d^{\mathcal{G}}(x,x') + d^{\mathcal{G}}(y,y')). \end{aligned}$$

- Likewise for $\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2} p_t^{\mathcal{G}}(\cdot,\cdot)$, using $p_t(\cdot,y) \in D(\Delta^{\mathcal{G}})$ for a.e. $y \in \mathcal{G}$.

# More general operators

## Proposition

*Everything we have seen is still valid if* $\Delta$ *is replaced by*

$$A_{c,V,\gamma}u := \frac{\partial}{\partial x}\left(c(\cdot)\frac{\partial}{\partial x}\right) + V$$

*with "$\delta$-interaction"*

$$\text{continuity} \quad + \quad \sum_{\mathsf{e}\sim\mathsf{v}} c_{\mathsf{e}}(\mathsf{v})\frac{\partial u_{\mathsf{e}}}{\partial n}(\mathsf{v}) + \gamma(\mathsf{v})u(\mathsf{v}) = 0$$

*for* $c \in L^\infty(\mathcal{G})$, $V \in L^1(\mathcal{G})$, *and* $(\gamma(\mathsf{v}))_{\mathsf{v}\in\mathsf{V}}$.

## Proof.

$A_{c,V,\gamma}$ is associated with

$$\mathfrak{a}^{\mathcal{G}}_{c,V,\gamma}(f) := \int_{\mathcal{G}} a(x)|f'(x)|^2\,\mathrm{d}x + \int_{\mathcal{G}} V(x)|f(x)|^2\,\mathrm{d}x + \sum_{\mathsf{v}\in\mathsf{V}} \gamma(\mathsf{v})|f(\mathsf{v})|^2$$

with same form domain $D(\mathfrak{a}^{\mathcal{G}}_{c,V,\gamma}(f)) = D(\mathfrak{a}^{\mathcal{G}}) = H^1(\mathcal{G})$. □

! Dirichlet conditions at a vertex can be obtained letting $\gamma(\mathsf{v}) \to +\infty$.

# Lack of domination

## Proposition

*If $\mathcal{G}, \mathcal{G}'$ any two different wirings over the same edge set, then $\mathrm{e}^{t\Delta^{\mathcal{G}}}$ does not dominate $\mathrm{e}^{t\Delta^{\mathcal{G}'}}$ for any $t > 0$.*

## Proof.

$D(\mathfrak{a}^{\mathcal{G}})$ is not a generalized ideal of $D(\mathfrak{a}^{\mathcal{G}'})$ (Exercise) □

# Miscellaneous comments

- Kennedy–Lang 2020: Similar results also hold operators with $V \in L^1(\mathcal{G};\mathbb{C})$, $(\gamma(\mathsf{v}))_{\mathsf{v}\in\mathsf{V}} \subset \mathbb{C}$. In particular, $|\mathrm{e}^{tA_{c,V,\gamma}}| \le \mathrm{e}^{tA_{c,\mathrm{Re}\,V,\mathrm{Re}\,\gamma}}$
- Kurasov 2010, Berkolaiko–Weyand 2012, Egidi–M.–Seelmann 2023: One can also add a magnetic potential: somewhat trivial, because a gauge transformation makes $\Delta_\alpha$ similar to $\Delta$. A diamagnetic inequality holds:

$$|\mathrm{e}^{t\Delta_\alpha}| \le \mathrm{e}^{t\Delta} \qquad \text{for all } t > 0.$$

- Glück–M. 2021: If $\mathcal{G}, \mathcal{G}'$ any two different wirings over the same edge set, then $\mathrm{e}^{t\Delta^{\mathcal{G}}}$ does not even *eventually* dominate $\mathrm{e}^{t\Delta^{\mathcal{G}'}}$:

  there is no $t_0 > 0$ such that $\mathrm{e}^{t\Delta^{\mathcal{G}}} \le \mathrm{e}^{t\Delta^{\mathcal{G}'}}$ for all $t > t_0$.

  Open question: Given two different wirings $\mathcal{G}, \mathcal{G}'$, is there $M > 0$ such that $\mathrm{e}^{t\Delta^{\mathcal{G}}} \le M\mathrm{e}^{t\Delta^{\mathcal{G}'}}$ for all $t > 0$?

# Long-time behavior

By resolvent compactness, $\Delta_{\mathcal{G}}$ has an ONB of eigenfunctions $(\varphi_k)$ with associated eigenvalues $-\lambda_k = -\lambda_k(\mathcal{G})^2$.

If $\mathcal{G}$ is connected, then $\lambda_0 = 0$ (simple!) with $\varphi_0 = \mathbf{1}_{\mathcal{G}}$.

Because $\mathrm{e}^{t\Delta^{\mathcal{G}}} f(\cdot) = \sum_{k=0}^{\infty} \mathrm{e}^{-t\lambda_k} \varphi_k(\cdot) \int_{\mathcal{G}} \varphi_k(x) f(x)\,\mathrm{d}x$,

$$\begin{aligned}\|\mathrm{e}^{t\Delta^{\mathcal{G}}} f - \int_{\mathcal{G}} \varphi_0(x) f(x)\,\mathrm{d}x\| &= \|\sum_{k=1}^{\infty} \mathrm{e}^{-t\lambda_k} \varphi_k \int_{\mathcal{G}} \varphi_k(x) f(x)\,\mathrm{d}x\| \\ &\leq \mathrm{e}^{-t\lambda_1} \|f\|\end{aligned}$$

Estimating $\lambda_1$ is crucial to study the long-time behaviour!

[2] Recall: $\lambda_k \geq 0$!

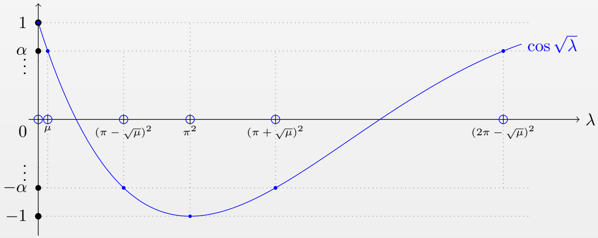
# The Laplacian on metric graphs and their underlying combinatorial graphs

Given $\mathcal{G}$, consider the underlying combinatorial graph G, its degree matrix $\mathcal{D}^{\mathsf{G}}$ and its discrete Laplacian $\mathcal{L}^{\mathsf{G}}$.

## Proposition (von Below 1985)

*If all $\ell_e \equiv \ell$, TFAE:*

- *$\lambda$ is eigenvalue of $-\Delta^{\mathcal{G}}$*
- *$\alpha := \cos\sqrt{\lambda}$ is eigenvalue of $\mathrm{Id} - \mathcal{D}^{\mathsf{G}^{-\frac{1}{2}}}\mathcal{L}^{\mathsf{G}}\mathcal{D}^{\mathsf{G}^{-\frac{1}{2}}}$*



# Nicaise' Isoperimetric Inequality

## Theorem (Nicaise 1987)

*For any metric graph $\mathcal{G}$ on finitely many edges of finite length* $\lambda_1(\mathcal{G}) \geq \frac{\pi^2}{|\mathcal{G}|^2}$, *with equality if* $\mathcal{G}=$ 

## Theorem (Friedlander 2005)

*Nicaise' Inequality is sharp. Indeed*

$$\lambda_j(\Delta_{\mathcal{G}}) \geq \frac{\pi^2 (j+1)^2}{4|\mathcal{G}|^2} \qquad \text{for all } j \in \mathbb{N},$$

*with equality if (and only if!)* $\mathcal{G}$ *is a metric star on* $j+1$ *edges of same length.*

## Exercise (Nicaise 1987)

*Prove the estimate* $\lambda_1(\mathcal{G}; \mathrm{V}^{\mathrm{D}}) \geq \frac{\pi^2}{4|\mathcal{G}|^2}$ *if* $\mathrm{V}^{\mathrm{D}} \neq \emptyset$, *with equality iff* $\mathcal{G}=$ .

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# Proof of Nicaise' Inequality – Kurasov–Naboko's version

- Produce $\mathcal{G}_{(2)}$ by replacing each edge e in $\mathcal{G}$ by two identical copies of e: then $|\mathcal{G}_{(2)}| = 2|\mathcal{G}|$.
- Take $(\lambda_1, \varphi_1)$ and clone $\varphi_1$ to produce an admissibile test function $\varphi_1^{(2)}$ for $\lambda_1(\mathcal{G}_{(2)})$: observe that $\varphi_1^{(2)} \perp \mathbf{1}_{\mathcal{G}_{(2)}}$.
- Also, $\|\varphi_1^{(2)}\|_{L^2}^2 = 2\|\varphi_1\|_{L^2}^2$, $\|\varphi_1^{(2)'}\|_{L^2}^2 = 2\|\varphi_1'\|_{L^2}^2$: hence $\lambda_1(\mathcal{G}) = \frac{\|\varphi_1'\|_{L^2}^2}{\|\varphi_1\|_{L^2}^2} \geq \min\limits_{\substack{f \in H^1(\mathcal{G}_{(2)}) \\ f \perp \mathbf{1}_{\mathcal{G}_{(2)}}}} \frac{\|f'\|_{L^2}^2}{\|f\|_{L^2}^2} = \lambda_1(\mathcal{G}_{(2)})$.
- Cut through all vertices to turn $\mathcal{G}_{(2)}$ into a cycle $\mathcal{C}$: this is possible because each vertex in $\mathcal{G}_{(2)}$ has even degree, so $\mathcal{G}_{(2)}$ contains a Eulerian cycle: $\lambda_1(\mathcal{G}_{(2)} \geq \lambda_1(\mathcal{C})$.
- However, $\lambda_1(\mathcal{C}) = \frac{4\pi^2}{|\mathcal{C}|^2} = \frac{\pi^2}{|\mathcal{G}|^2}$.

# Selected surgery principles

## Proposition (Kennedy–Kurasov–Malenová–M. 2016)

*Given $\mathcal{G}$ with finitely many edges of finite length, produce $\mathcal{G}'$ by*

1. *cutting through a vertex* $\mathsf{v}$ *to create two new vertices* $\mathsf{v}_1, \mathsf{v}_2 \in \mathcal{G}$, *or*
2. *attaching a pendant graph* $\mathcal{H}$ *at a vertex* $\mathsf{v} \in \mathcal{G}$.

*Then* $\lambda_k(\mathcal{G}) \geq \lambda_k(\mathcal{G}')$.
*Furthermore,* $\lambda_1(\mathcal{G}) = \lambda_1(\mathcal{C})$ *if*

3. $\mathcal{G}$ *is a figure-8 graph and* $\mathcal{C}$ *is a cycle graph with* $|\mathcal{G}| = |\mathcal{C}|$.

## Proof.

(1) $H^1(\mathcal{G}) \supset H^1(\mathcal{G}')$
(2) Take $(\lambda_1, \varphi_1)$ and extend $\varphi_1$ by continuity to a function that is constant on $\mathcal{H}$. Then $\varphi_1 \mathbf{1}_{\mathcal{G}} - |\mathcal{H}| \mathbf{1}_{\mathcal{H}}$ is orthogonal to $\mathbf{1}_{\mathcal{G}'}$, hence an admissible test function for $\lambda_1(\mathcal{G}')$.
(3) Construct $\mathcal{C}$ from $\mathcal{G}$ by cutting through the (only) vertex $\mathsf{v}$, thus creating $\mathsf{v}_1, \mathsf{v}_2$. By (1), $\lambda_1(\mathcal{C}) \leq \lambda_1(\mathcal{G})$.
Pick a ground state $\psi_1$ on $\mathcal{C}$: up to rotation, wlog $\psi_1(\mathsf{v}_1) = \psi_1(\mathsf{v}_2)$: thus, $\psi_1 \in H^1(\mathcal{G})$ is an admissible test function on $\mathcal{G}$, hence $\lambda_1(\mathcal{G}) \leq \lambda_1(\mathcal{C})$. □

# An upper estimate

**Theorem (Kennedy–Kurasov–Malenová–M. 2016)**

*For any metric graph $\mathcal{G}$ on $E \geq 2$ edges of finite length*

$$\lambda_1(\mathcal{G}) \leq \frac{\pi^2 E^2}{|\mathcal{G}|^2}.$$

*Equality holds for equilateral stars and equilateral pumpkin graphs...*

M.–Pivovarchik 2022: ...and for an infinite class of metric graphs ("inflated stars", after Butler–Grout 2011).

# Proof

- Glue *all* vertices to produce a new metric graph $\mathcal{G}'$ (a "metric flower"): then $\lambda_1(\mathcal{G}) \leq \lambda_1(\mathcal{G}')$.
- Produce a figure-8 graph $\mathcal{G}''$ by plucking all petals of the metric flower but the two longest ones: then $\lambda_j(\mathcal{G}') \leq \lambda_j(\mathcal{G}'')$ for all $j$.
- $\lambda_1(\mathcal{G}'') = \lambda_1(\text{Cycle of same total length as } \mathcal{G}') = \frac{4\pi^2}{|\mathcal{G}''|^2}$ (easy proof using symmetry).
- However, by the pigeonhole principle $|\mathcal{G}''| \geq 2\frac{|\mathcal{G}|}{E}$.

# Weyl asymptotics

Recall:

$$\lambda_j(\Delta_{\mathcal{G}}) \geq \frac{\pi^2(j+1)^2}{4|\mathcal{G}|^2} \qquad \text{for all } j \in \mathbb{N},$$

## Proposition

*Given $\mathcal{G}$ on $E < \infty$ edges of finite length,*

$$\lambda_j(\mathcal{G}) \leq \frac{E^2\pi^2(j+1)^2}{|\mathcal{G}|^2}$$

## Proof.

Repeat the previous proof and, in the last step, observe that $\lambda_j(\mathcal{G}'') \leq \lambda_{j+1}(\text{Cycle of same total length as } \mathcal{G}') \leq \frac{(j+1)^2\pi^2}{|\mathcal{G}''|^2}$ (again by symmetry). □

## Corollary (Nicaise 1987)

$$\frac{\lambda_j(\mathcal{G})}{j^2} \approx \frac{\pi^2}{|\mathcal{G}|^2}$$

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$$\frac{\lambda_j(\mathcal{G})}{j^2} \approx \frac{\pi^2}{|\mathcal{G}|^2}$$

# Eigenvalue estimates with Dirichlet vertex conditions

## Proposition (Plümer 2022)

*If $\mathcal{G}$ is a graph with finitely many edges of finite length, then*

$$\lambda_1(\mathcal{G};\mathrm{V}^{\mathrm{D}}) \geq \frac{1}{|\mathcal{G}|\,\mathsf{Inr}(\mathcal{G};\mathrm{V}^{\mathrm{D}})}$$

*where* $\mathsf{Inr}(\mathcal{G};\mathrm{V}^{\mathrm{D}}) := \sup_{x\in\mathcal{G}} d(x,\mathrm{V}^{\mathrm{D}})$.

## Proof.

Let $f \in H_0^1(\mathcal{G};\mathrm{V}^{\mathrm{D}})$, $x \in \mathcal{G}$, $\mathsf{v} \in \mathrm{V}^{\mathrm{D}}$, $\gamma$ a geodesic between $x, \mathsf{v}$. Then

$$f(x) = f(x) - f(\mathsf{v}) = \int_\gamma f'(y)\,\mathrm{d}y$$

and

$$|f(x)|^2 \leq L(\gamma)\int_\gamma |f'(y)|^2\,\mathrm{d}y \leq d(x,\mathrm{V}^{\mathrm{D}})\|f'\|^2_{L^2(\mathcal{G})}.$$

Therefore,

$$\|f\|^2_{L^2(\mathcal{G})} \leq \int_\mathcal{G} d(x,\mathrm{V}^{\mathrm{D}})\,\mathrm{d}x\|f'\|^2_{L^2(\mathcal{G})} = |\mathcal{G}| \fint_\mathcal{G} d(x,\mathrm{V}^{\mathrm{D}})\,\mathrm{d}x\|f'\|^2_{L^2(\mathcal{G})}$$

$$\leq |\mathcal{G}|\,\mathsf{Inr}(\mathcal{G};\mathrm{V}^{\mathrm{D}})\|f'\|^2_{L^2(\mathcal{G})}.$$

# Lower estimate by diameter and nodal counting

## Corollary

*If $\mathcal{G}$ is a graph with finitely many edges of finite length, then*

$$\lambda_k(\mathcal{G}) \geq \frac{\nu_k}{|\mathcal{G}|\,\mathrm{Diam}(\mathcal{G})},$$

*where $\nu_k$ is # of nodal domains $\mathcal{G}_1, \ldots, \mathcal{G}_k$ of $\psi_k$; in particular,*

$$\lambda_1(\mathcal{G}) \geq \frac{2}{|\mathcal{G}|\,\mathrm{Diam}(\mathcal{G})}.$$

## Proof.

$\lambda_k(\mathcal{G}) = \lambda_1(\mathcal{G}_i; \partial\mathcal{G}_i)$, hence

$$\lambda_k(\mathcal{G}) \geq \frac{1}{|\mathcal{G}_j|\,\mathrm{Inr}(\mathcal{G}_j; \partial\mathcal{G}_j)}.$$

By the pidgeonhole principle, there is $j$ with $|\mathcal{G}_j| \leq \frac{|\mathcal{G}|}{\nu_k}$. □

# Lower estimate by mean distance

## Corollary (Baptista–Kennedy–M. 2023)

*If $\mathcal{G}$ is a graph with finitely many edges of finite length, then*

$$\lambda_1(\mathcal{G}) \geq \frac{1}{|\mathcal{G}| ⨍_{\mathcal{G}\times\mathcal{G}} d(x,y)\,\mathrm{d}x\,\mathrm{d}y}.$$

## Proof.

- Pick $x_0 \in \mathcal{G}$ with $⨍_{\mathcal{G}} d(x_0,y)\,\mathrm{d}y = ⨍_{\mathcal{G}\times\mathcal{G}} d(x,y)\,\mathrm{d}x\,\mathrm{d}y$.
- Use Plümer's estimate to deduce (for $\mathrm{V}^{\mathrm{D}} := \{x_0\}$)

$$1 \leq \lambda_1(\mathcal{G};\{x_0\})|\mathcal{G}| ⨍_{\mathcal{G}\times\mathcal{G}} d(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

- Consider the nodal domains $\mathcal{G}_\pm$ of $\Delta^{\mathcal{G}}$, assume wlog that $x_0 \in \mathcal{G}_+$ and deduce from domain monotonicity of the Dirichlet eigenvalues that

$$\lambda_1(\mathcal{G}) = \lambda_1(\mathcal{G}_+;\partial\mathcal{G}_+) \geq \lambda_1(\mathcal{G};\{x_0\})$$

□

# Lower estimate by avoidance diameter

The **avoidance diameter** of $\mathcal{G}$ is

$$\mathrm{avoid}(\mathcal{G}) := \max_{\gamma} \min_{x \in \mathbb{S}^1} d(\gamma(-x), \gamma(x))$$

where max is taken over all injective continuous $\gamma : \mathbb{S}^1 \to \mathcal{G}$.

| $\mathcal{G}$ | $\mathrm{avoid}(\mathcal{G})$ |
|---|---|
| trees | 0 |
| equilateral figure-8 graph | $\frac{L}{4}$ |
| equilateral flower graph on $k$ edges | $\frac{L}{2k}$ |
| equilateral pumpkin graph on $k$ edges | $\frac{L}{k}$ |

Proposition (Berkolaiko–Kennedy–Kurasov–M. 2023:)

$$\lambda_1(\mathcal{G}) < \frac{6|\mathcal{G}|}{\mathrm{avoid}(\mathcal{G})^3}$$

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# A homotopy lemma

## Lemma

*Let $\mathfrak{A}$ be closed quadratic form with dom$(\mathfrak{A}) \overset{c}{\hookrightarrow} L^2(X;\mu)$. Assume the associated operator $A$ on $L^2(X;\mu)$ to have one-dimensional null space spanned by some $u$.*
*If*

- $\psi_\cdot : [0,1] \to D(a) \setminus \{0\}$ *satisfies* $\psi_0 = -\psi_1$ *and*
- $[0,1] \ni t \mapsto (\psi_t, u) \in \mathbb{R}$ *is continuous*

*then the second lowest eigenvalue $\lambda_1(A)$ of $A$ satisfies*

$$\lambda_1(A) \leq \frac{\mathfrak{A}(\psi_{t_0})}{\|\psi_{t_0}\|_{L^2}^2} \qquad \text{for some } t_0 \in (0,1).$$

In our relevant case: $\mathfrak{A}(f) = \int_{\mathcal{G}} |f'|^2 \,\mathrm{d}x$, $u \equiv 1$.

Proof.

Because $(\psi_0, u) = -(\psi_1, u)$, there is $t_0$ with $(\psi_{t_0}, u) = 0$. Now, use $\psi_{t_0}$ as a test function in the Rayleigh quotient. $\square$

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# Sketch of the proof

Apply the homotopy lemma to

$$\psi_t := \tau_{\gamma(e^{i\pi t}),\frac{1}{2}\mathrm{avoid}(\mathcal{G})} - \tau_{\gamma(-e^{i\pi t}),\frac{1}{2}\mathrm{avoid}(\mathcal{G})}, \qquad t \in [0, 2\pi),$$

where $\gamma$ is the curve realizing the avoidance diameter and

$$\tau_{x,d}(y) := \begin{cases} d - d(x,y), & \text{if } d(x,y) \le d, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\lambda_1(\mathcal{G}) \le \max_{t\in[0,1]} \frac{|\mathcal{G}|}{2\|\tau_{\gamma(e^{i\pi t}),\mathrm{avoid}(\mathcal{G})}\|^2} \le \frac{6|\mathcal{G}|}{\mathrm{avoid}(\mathcal{G})^3}.$$

1. Heat equation and heat kernels

2. Laplacians on metric graphs

3. Spectral geometry
   - Basic estimates in terms of total length
   - Alternative estimates using different quantities

4. Thermal geometry

5. Nonlinear diffusion

# Shape optimization wrt heat kernel?

Already seen:

If $\mathcal{G}, \mathcal{G}'$ are two different wirings over the same edge set,

$$p_t^{\mathcal{G}}(x,y) \leq p_t^{\mathcal{G}'}(x,y) \qquad \forall x, y \in \mathcal{G}$$

for all $t \geq 0$ is impossible.

Idea: Consider the **overall insulation** wrt $\mathrm{V^D}$

$$\int_0^\infty \int_{\mathcal{G}} \int_{\mathcal{G}} p_t^{\mathcal{G};\mathrm{V^D}}(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

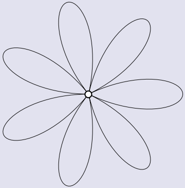
## Remark

- Because $p_t^{\mathcal{G}} \geq 0$, so is $\int_0^\infty \int_{\mathcal{G}} \int_{\mathcal{G}} p_t^{\mathcal{G}}(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$.
- The Green function $G^{\mathcal{G};\mathrm{V^D}}$ is the Laplace transform of $p_\cdot^{\mathcal{G};\mathrm{V^D}}$ (Exercise).
- If $\mathrm{V^D} = \emptyset$, the overall insulation is always $= \infty$, because $\int_{\mathcal{G}} \int_{\mathcal{G}} p_t^{\mathcal{G};\mathrm{V^D}}(x,y)\,\mathrm{d}x\,\mathrm{d}y = |\mathcal{G}|$ (Exercise).

# Path graphs maximize insulation

**Theorem**

$$\frac{1}{12}\frac{|\mathcal{G}|^3}{|\mathsf{E}|^3} \le \int_0^\infty \int_{\mathcal{G}} \int_{\mathcal{G}} p_t^{\mathcal{G}}(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \le \frac{1}{3}|\mathcal{G}|^3$$

*Lower estimate is an equality iff* $\mathcal{G} =$ 

*Upper estimate is an equality iff* $\mathcal{G} =$ 

# Proof (upper estimate)

- $\int_0^\infty p_t^{\mathcal{G}}(x,y)\,\mathrm{d}t$ is the Green's function of $\mathcal{G}$, i.e., the integral kernel of $\Delta^{-1}$.
- Thus, $\int_0^\infty \int_{\mathcal{G}} \int_{\mathcal{G}} p_t^{\mathcal{G}}(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = -\int_{\mathcal{G}} \Delta^{-1}\mathbb{1}(x)\,\mathrm{d}x$
- Describe the integrated heat content in variational terms, following Pólya:

$$-\int_{\mathcal{G}} (\Delta^{\mathcal{G};\mathrm{V}^{\mathrm{D}}})^{-1}\mathbb{1}(x)\,\mathrm{d}x = \max_{u\in H_0^1(\mathcal{G};\mathrm{V}^{\mathrm{D}})} \frac{\|u\|_{L^1}^2}{\|u'\|_{L^2}^2}$$

because the Euler–Lagrange equation for

$$-\Delta^{\mathcal{G};\mathrm{V}^{\mathrm{D}}} u = \mathbb{1}$$

is

$$\frac{1}{2}\int_{\mathcal{G}} u'(x)h'(x)\,\mathrm{d}x = \int_{\mathcal{G}} h(x)\,\mathrm{d}x, \qquad h \in H_0^1(\mathcal{G};\mathrm{V}^{\mathrm{D}})$$

- Mimic Nicaise' doubling trick.

# Proof (lower estimate)

- Use again $\int_0^\infty \int_{\mathcal{G}} \int_{\mathcal{G}} p_t^{\mathcal{G}}(x,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = \max_{u \in H_0^1(\mathcal{G};\mathrm{V}^{\mathrm{D}})} \frac{\|u\|_{L^1}^2}{\|u'\|_{L^2}^2}$
- Consider, as a test function, the function $u^*$ that satisfies $-{u^*}_{\mathrm{e}}'' = \mathbb{1}$ with Dirichlet conditions on each edge.
- Check that

$$\frac{\|u_{\mathrm{e}}^*\|_{L^1}^2}{\|{u^*}_{\mathrm{e}}'\|_{L^2}^2} = \frac{|\mathrm{e}|^3}{12}$$

and use Jensen.

# Landscape functions on metric graphs, after Filoche–Mayboroda

## Theorem

*Let* $\mathsf{V}^{\mathrm{D}} \neq \emptyset$. *Then each eigenpair* $(\lambda, \varphi)$ *of* $-\Delta^{\mathcal{G};\mathsf{V}^{\mathrm{D}}}$ *(even of the magnetic Laplacian* $\Delta_{\alpha}^{\mathcal{G};\mathsf{V}^{\mathrm{D}}}$*!) satisfies*

$$\frac{|\varphi(x)|}{\|\varphi\|_\infty} \leq \inf_{\delta>0} \delta \left[ \left( -\lambda_1 + \delta - \left( (-\Delta^{\mathcal{G};\mathsf{V}^{\mathrm{D}}})^{-1} \mathbb{1} \right) \right] (x)$$

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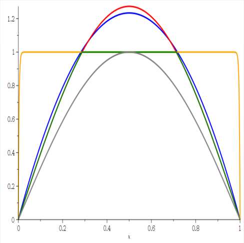


# Application to the heat kernel

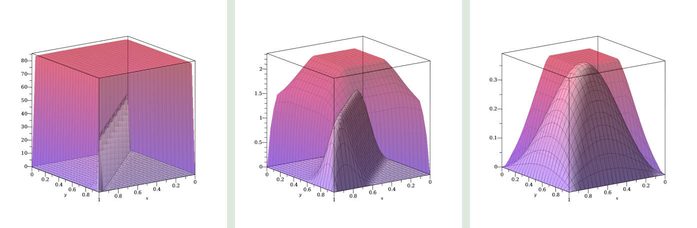
## Proposition

*There exists* $C = C(\mathcal{G})$ *with*

$$p_t^{\mathcal{G};\mathrm{VD}}(x,y) \leq C \left[ \sum_{k\in\mathbb{N}} |\lambda_k|^2 \mathrm{e}^{-t\lambda_k} \right] (-\Delta^{\mathcal{G};\mathrm{VD}})^{-1}\mathbb{1}(x)(-\Delta^{\mathcal{G};\mathrm{VD}})^{-1}\mathbb{1}(y).$$

## Example

If $\mathcal{G} =$ 



Same estimates holds even for the heat kernel of the magnetic Laplacian!

# Proof (for general magnetic Laplacians)

- Consider an ONB of eigenvectors of $\Delta^{\mathcal{G};\mathrm{V}^{\mathrm{D}}}$. Then

$$\varphi_k = \lambda_k(-\Delta_\alpha^{\mathcal{G};\mathrm{V}^{\mathrm{D}}})^{-1}\varphi_k$$

and because $\mathrm{e}^{t\Delta^{\mathcal{G};\mathrm{V}^{\mathrm{D}}}}$ dominates $\mathrm{e}^{t\Delta_\alpha^{\mathcal{G};\mathrm{V}^{\mathrm{D}}}}$

$$|\varphi_k| = |\lambda_k(-\Delta_\alpha^{\mathcal{G};\mathrm{V}^{\mathrm{D}}})^{-1}\varphi_k| \le |\lambda_k|(-\Delta^{\mathcal{G};\mathrm{V}^{\mathrm{D}}})^{-1}|\varphi_k| \le |\lambda_k|\|\varphi_k\|_\infty(-\Delta^{\mathcal{G};\mathrm{V}^{\mathrm{D}}})^{-1}\mathbb{1}.$$

- Bifulco–Kerner 2022: There exists $C(\mathcal{G})$ such that $\|\varphi_k\|_\infty \le C(\mathcal{G})$ for all $k$.
- By Mercer,

$$\begin{aligned} p_t^{\mathcal{G};\mathrm{V}^{\mathrm{D}}}(x,y) &= \sum_{k\in\mathbb{N}} \mathrm{e}^{-t\lambda_k}\varphi_k(x)\varphi_k(y) \\ &\le C(\mathcal{G})^2 \sum_{k\in\mathbb{N}} |\lambda_k|^2 \mathrm{e}^{-t\lambda_k}(-\Delta^{\mathcal{G};\mathrm{V}^{\mathrm{D}}})^{-1}\mathbb{1}(x)(-\Delta^{\mathcal{G};\mathrm{V}^{\mathrm{D}}})^{-1}\mathbb{1}(y). \end{aligned}$$

# Unlike eigenfunctions, the torsion function can be computed explicitly

## Exercise

*Let* $\mathcal{G}$ *be equilateral* ($\ell_{\mathsf{e}} \equiv 1$) *and let* $v := (-\Delta^{\mathcal{G};\mathsf{V}^{\mathrm{D}}})^{-1}\mathbb{1}$, *for* $\mathsf{V}^{\mathrm{D}} \neq \emptyset$. *Then the restriction* $g := v_{|\mathsf{V}} : \mathsf{V} \to \mathbb{R}$ *is the unique solution of the system*

$$\begin{cases} g(\mathsf{v}) = 0, & \mathsf{v} \in \mathsf{V}^{\mathrm{D}}, \\ \dfrac{1}{\deg(\mathsf{v})} \displaystyle\sum_{\mathsf{w}\sim\mathsf{v}} g(\mathsf{v}) - g(\mathsf{w}) = \dfrac{1}{2}, & \mathsf{v} \in \mathsf{V} \setminus \mathsf{V}^{\mathrm{D}}. \end{cases}$$

## Gradient of quadratic forms

Recall: given a closed quadratic form $\mathfrak{A}$ with corresponding bilinear form $\mathfrak{a}$, the associated operator $A$ satisfies

$$\mathfrak{a}(f,h) = (-Af,h) \quad \forall f \in D(A) \text{ and } h \in D(\mathfrak{a})\}$$

Indeed, $\mathfrak{A}$ is infinitely many times continuously differentiable, and in particular (Exercise)

$$\mathfrak{A}'(f)h = \mathfrak{a}(f,h) = (-Af,h)$$

. Then $-A$ is the **gradient** of $\mathfrak{A}$: $-A = \partial\mathfrak{A}$.

### Example

For the Dirichlet form $\mathfrak{A}(f) = \frac{1}{2}\int_\Omega |\nabla f(x)|^2 \, \mathrm{d}x$, $f \in H_0^1(\Omega)$, there holds $\mathfrak{A}'(f)g = \int_\Omega \nabla f(x) \nabla g(x) \, \mathrm{d}x = -\int_\Omega \Delta^{\Omega;\mathrm{D}} f(x) g(x) \, \mathrm{d}x$ : i.e., $\partial\mathfrak{A} = -\Delta^{\Omega;\mathrm{D}}$.

### Example

For all $p > 1$ $\mathfrak{A}_p(f) = \frac{1}{p}\int_\mathcal{G} |\nabla f(x)|^p \, \mathrm{d}x$, $f \in W_0^{1,p}(\Omega)$, is differentiable with derivative

$$\mathfrak{A}_p{}'(f)h = \int_\Omega |\nabla f(x)|^{p-2} \nabla f(x) \nabla g(x) \, \mathrm{d}x \stackrel{!}{=} -\int_\mathcal{G} \Delta_p^{\Omega;\mathrm{D}} f(x) h(x) \, \mathrm{d}x,$$

i.e., $-\partial\mathfrak{A}_p$ is the $p$-**Laplacian** on $\Omega$ (with Dirichlet BCs).

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# Nonlinear semigroups

**Theorem (Brezis 1973)**

*Given a convex, lsc, proper, coercive energy* $\mathfrak{A} : L^2(X;\mu) \to [0,\infty]$*, the Cauchy problem for*

$$\partial_t u + \partial\mathfrak{A}(u) = 0$$

*is well-posed.*

$\rightsquigarrow$ for all initial data $u_0$ there exists a solution

$$t \mapsto u(t) =: \mathrm{e}^{-\partial\mathfrak{A}} u_0$$

# Long time behavior vs $p$-homogeneity

Let $\mathfrak{A}_p$ be convex, lsc, proper, coercive and $p$-homogeneous:

- we can consider
$$\lambda_{1,p} := \inf_{u \perp \mathrm{Ker}(\mathfrak{A}_p)} \frac{p\mathfrak{A}(u)}{\|u\|^p} > 0;$$
- $u \perp \mathrm{Ker}(\mathfrak{A}_p)$ is called **eigenfunction** of $\partial\mathfrak{A}$ for the *(variational) eigenvalue* $\lambda$ if
$$\lambda\|u\|^{p-2}u = \partial\mathfrak{A}_p(u).$$

## Convergence to steady state vs *p*-homogeneity

If $p = 2$,

$$\|u(t)\|^2 \leq \|u(0)\|^2 \mathrm{e}^{-2\lambda_1 t}$$

**Theorem (Bungert–Burger 2020)**

*Let $\mathfrak{A}_p$ be convex, lsc, proper, coercive and p-homogeneous, for $p \geq 1$: then the solution of $\partial_t u + \partial \mathfrak{A}_p(u) = 0$ with $u(0) \perp \mathrm{Ker}(\mathfrak{A}_p)$ satisfies*

$$\|u(t)\|^2 \leq \frac{1}{\|u(0)\|^{2-p} + (p-2)\lambda_{1,p} t} \qquad \text{if } p \in (2, \infty),$$

$$\|u(t)\|^2 \leq \|u(0)\|^{2-p} - (2-p)\lambda_{1,p} t \qquad \text{if } p \in [1, 2).$$

**Remark**

In particular, for $p < 2$

$$\|u(t)\|^{2-p} \geq (2-p)\lambda_{1,p}(T_{ex} - t) \quad \text{and} \quad u(t) \equiv 0 \quad \forall t \geq T_{ex},$$

where

$$T_{ex} \leq \frac{\|u(0)\|^{2-p}}{(2-p)\lambda_{1,p}} < \infty.$$

Likewise: infinite extinction time if $p > 2$.

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Estimating $\lambda_{1,p} \equiv$ controlling long-time behavior. For $\Omega$ and $p \neq 2$:

- Dirichlet: Bhattacharya (1999)
- Neumann: Brasco–Nitsch–Trombetti (2016)

## Prototypical Example

*The p-Laplacian $\Delta_p^{\mathcal{G}}$ is (minus) the derivative $-\mathfrak{A}^{\mathcal{G}}{}_p$ in $L^2(\mathcal{G})$ of the energy*

$$\mathfrak{A}_p(f) := \frac{1}{p}\int_{\mathcal{G}} |f'|^p \,\mathrm{d}x, \qquad f \in W^{1,p}(\mathcal{G}) := C(\mathcal{G}) \cap \bigoplus_{\mathsf{e}\in\mathsf{E}} W^{1,p}(0,\ell_{\mathsf{e}}).$$

*($\rightsquigarrow$ continuity + nonlinear Kirchhoff-type vertex conditions)*

$\Delta_p^{\mathcal{G}}$ *generates on $L^2(\mathcal{G})$ a nonlinear (Markovian) semigroup.*

*(Likewise if Dirichlet conditions are imposed on a vertex subset $\mathsf{V}^{\mathrm{D}} \subset \mathsf{V}$.)*

**Proposition (Hofmann–Kennedy–M.–Plümer 2021)**

$-\Delta_p^{\mathcal{G}}$ *has countably many eigenvalues* $0 = \lambda_{0,p}(\mathcal{G}) \leq \lambda_{1,p}(\mathcal{G}) \leq \ldots \to +\infty$*:*

$$\lambda_{n,p}(\mathcal{G}) = (p-1)\left(\frac{\pi_p}{|\mathcal{G}|}\right)^p n^p + o(n^p) \qquad \text{as } n \to \infty,$$

*where* $\pi_p := \frac{2\pi}{p\sin(\frac{\pi}{p})}$, $p \in (1,\infty)$.

Bungert–Burger $\Rightarrow$ If $u(0) \perp \mathbb{1}$, then

$$\|u(t)\|^2 \leq \frac{1}{\|u(0)\|^{2-p} + (p-2)\lambda_{1,p} t} \qquad \text{if } p \in (2, \infty),$$

$$\|u(t)\|^2 \leq \|u(0)\|^{2-p} - (2-p)\lambda_{1,p} t \qquad \text{if } p \in [1, 2).$$

How fast/slow can convergence be?

**Theorem (Del Pezzo–Rossi 2016; Berkolaiko–Kennedy–Kurasov–M. 2017)**

*Given a graph on $E < \infty$ edges of finite length, for all $p \in (1, \infty)$*
$(p-1)\frac{\pi_p^p}{|\mathcal{G}|^p} \leq \lambda_{1,p}(\mathcal{G}) \leq (p-1)\frac{E^p \pi_p^p}{|\mathcal{G}|^p}$, *with equality iff* $\mathcal{G} =$ 

*If additionally $\mathcal{G}$ is 2-connected:*

$\lambda_{1,p}(\mathcal{G}) \geq 2^p (p-1)\frac{\pi_p^p}{|\mathcal{G}|^p}$, *with equality iff* $\mathcal{G} =$ 

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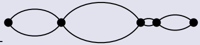
$\lambda_{1,p}(\mathcal{G}) \geq 2^p (p-1)\frac{\pi_p^p}{|\mathcal{G}|^p}$, *with equality iff $\mathcal{G} =$* 

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**Thank you for your attention!**