Guided quantum dynamics

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Quantum waveguides

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In the absence of other forces, the Hamiltonian is then the (negative) *Laplacian*, $-\Delta$, and the spectral problem means to solve the equation

$$-\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big)\psi(x,y)=\lambda\psi(x,y), \quad x\in\mathbb{R},\ |y|<a,$$

with *Dirichlet boundary condition* describing the hard wall, that is

$$\psi(x,\pm a)=0.$$

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To be specific, consider a curve $\Gamma: \mathbb{R} \to \mathbb{R}^2$ assuming that it is *smooth* and *asymptotically straight* and put $\Omega := \{x \in \mathbb{R}^2 : \mathrm{dist}(x, \Gamma) < a\}$; the strip considered above, which we denote as Ω_0, refers naturally to the trivial situation when Γ is a straight line.

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$$x(s,u) = \big(\Gamma_1(s) - u\dot\Gamma_2(s),\, \Gamma_2(s) + u\dot\Gamma_1(s)\big), \quad |u| < a.$$

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We transform $-\Delta$ into these coordinates and remove the Jacobian replacing, with an abuse of notation, $\psi(x)$ with $(1+u\gamma(s))^{1/2}\psi(s,u)$, where $\gamma(s) := (\ddot\Gamma_2\dot\Gamma_1 - \ddot\Gamma_1\dot\Gamma_2)(s)$ is the *signed curvature* of Γ; then we have to find the spectrum of the following Dirichlet operator in $L^2(\Omega_0)$:

$$H = -\frac{\partial}{\partial s}(1+u\gamma(s))^{-2}\frac{\partial}{\partial s} - \frac{\partial^2}{\partial u^2} + V(s,u),$$

$$V(s,u) := -\frac{\gamma(s)^2}{4(1+u\gamma(s))^2} + \frac{u\ddot\gamma(s)}{2(1+u\gamma(s))^3} - \frac{5}{4}\,\frac{u^2\dot\gamma(s)^2}{(1+u\gamma(s))^4}.$$

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$$H = -\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial s^2} - \frac{1}{4}\gamma(s)^2 + \mathcal{O}(a) \quad \text{as} \;\; a \to 0,$$

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Moral: Listen to your supervisor, but think twice before taking his advice!

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But we can do better, without restriction on the strip width. Consider any $a > 0$ for which the strip boundary is still smooth, $a\|\gamma\|_\infty < 1$, and the strip *does not intersect itself*.

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Using the Ansatz $\psi(s,u) = \phi_\lambda(s)\chi_1(u) + \varepsilon f(s,u)$, one can check that choosing appropriately functions $\phi_\lambda(s)$ and f and the number ε, we achieve the goal obtaining the following result:

Theorem

If the strip axis is a C^4 smooth curve, not straight but asymptotically straight [leaving out the precise formulation], the the Dirichlet Laplacian in the curved strip has at least one isolated eigenvalue below κ_1^2.

J. Goldstone, R.L. Jaffe: Bound states in twisting tubes, *Phys. Rev.* **B45** (1992), 14100–14107.

P. Duclos, P.E.: Curvature–induced bound states in quantum waveguides in two and three dimensions, *Rev. Math. Phys.* **7** (1995), 73–102.

How it differs from the classical motion?

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The comparison is only partly fitting, of course, one can note that a bobsleigh in a rectangular-shaped track would climb nowhere.

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To illustrate this claim, consider Ω in the shape of an *L-shaped strip*; we choose the width $2a = \pi$ so that $\kappa_1^2 = 1$

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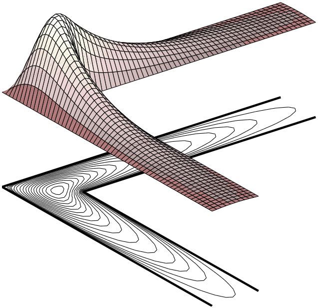
To illustrate this claim, consider Ω in the shape of an *L-shaped strip*; we choose the width $2a = \pi$ so that $\kappa_1^2 = 1$. Expanding the sought solution to $-\Delta_{\mathrm{D}}^{\Omega}\psi = \lambda\psi$ into the 'transverse' basis, one can prove that the operator has a single eigenvalue ≈ 0.929; the corresponding eigenfunction is



P.E., P. Šeba, P. Šťovíček: On existence of a bound state in an L-shaped waveguide, *Czech. J. Phys.* **B39** (1989), 1181–1191.

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Moreover, the binding effect coming from the geometry of the guide is *not restricted to bends*. For instance, it is not difficult to see that bound states occur if the tube has a local *'bulge'*.

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The *essential* (absolutely continuous) *spectrum* of the Hamiltonian H starts now at $\left(\frac{\pi}{d}\right)^2$, where $d = \max\{d_1, d_2\}$ and we have

Theorem

The discrete spectrum of H is nonempty for any $a > 0$ and

$$\sharp\sigma_{\mathrm{disc}}(H) \geq \frac{2a}{d}\sqrt{1 - \left(\frac{d}{d_1 + d_2}\right)^2}$$

P.E., P. Šeba, M. Tater, D. Vaněk: Bound states and scattering in quantum waveguides coupled laterally through a boundary window, *J. Math. Phys.* **37** (1996), 4867–4887.

Example: two particular cases

Let us plot two eigenfunction, the *ground state* for $d_1 = d_2$ and the *second excited state* is an asymmetric waveguide:

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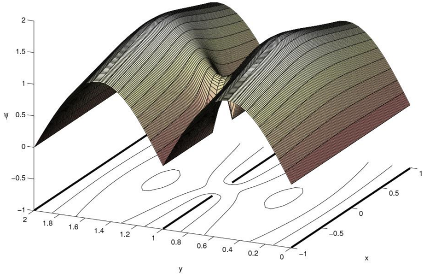

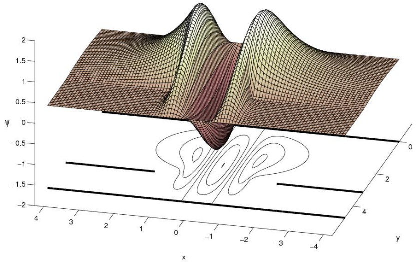

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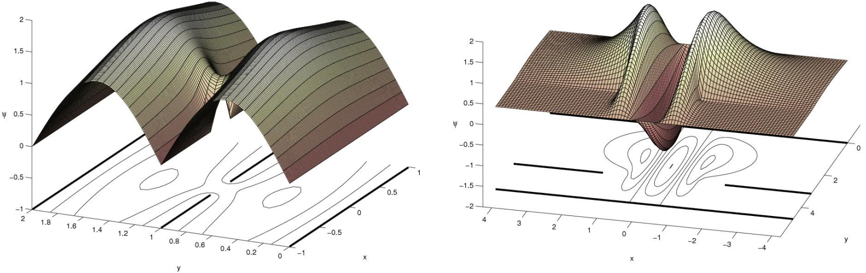

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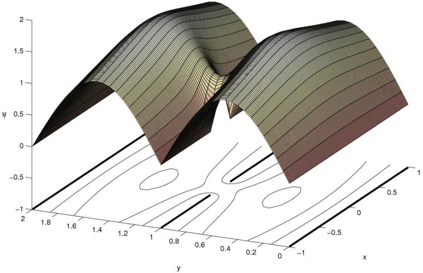

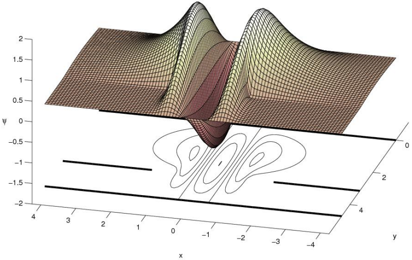

In particular, this example illustrates well the *purely quantum nature* of the effect: a classical particle in such a system *cannot be trapped* except for the (*phase-space measure zero!*) events of reflections, either from the window edges or perpendicular to the walls.

A detour: Šeba billiard

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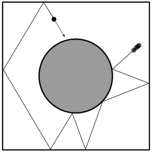

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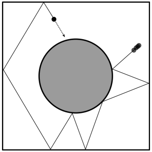

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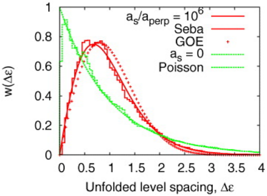

Source: [SAYO'10]

P. Šeba: Wave chaos in singular quantum billiard, *Phys. Rev. Lett.* **64** (1990), 1855–1858.

C. Stone, Y.A. El Aoudi, V.A. Yurovsky, M. Olshanii1: Two simple systems with cold atoms: quantum chaos tests and non-equilibrium dynamics, *New J. Phys.* **12** (2010), 055022.

More results about waveguides

- The results can be tested experimentally in *flat electromagnetic waveguides*.

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- Similar results hold for three-dimensional bent tubes of *circular cross section*.
- If the cross section *is not circular*, we have to consider the *twisting* which, in contrast to the bending, produces a *repulsive* interaction.

For many more results see

P.E., H. Kovařík: *Quantum Waveguides*; *xxii* + 382 p.; Springer International, Heidelberg 2015.

Quantum layers

If we take this exercise *one dimension higher*, we can observe other interesting phenomena

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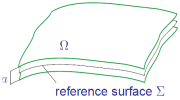

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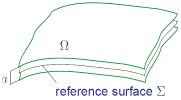

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$$H = -\frac{\partial^2}{\partial u^2} - g^{-1/2}\frac{\partial}{\partial s_\mu} g^{1/2} g^{\mu\nu} \frac{\partial}{\partial s_\nu} + K - M^2 + \mathcal{O}(a),$$

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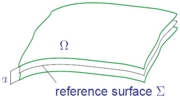

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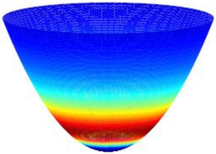
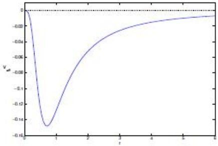
where g is *metric tensor* of the surface Σ, and K, M are its *Gauss* and *mean* curvatures, respectively. Since $K = k_1 k_2$ and $M = \frac{1}{2}(k_1 + k_2)$, the leading term of the effective potential, $K - M^2 = -\frac{1}{4}(k_1 - k_2)^2$, is again of the *attractive* nature, vanishing only on *planes* and *spheres*.

The effective potential in a thin layer

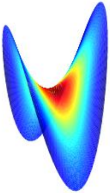
Effective Potential $\quad V_{\text{eff}} = -\frac{1}{4}(k_+ - k_-)^2$

Paraboloid of Revolution $\quad z = x^2 + y^2$

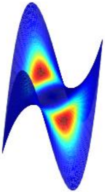




Hyperbolic Paraboloid $\quad z = x^2 - y^2$



Monkey Saddle $\quad z = x^3 - 3xy^2$



The minima of V_{eff} are marked by the dark red colour.

Curvature induced bound states in layers

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Theorem

If the surface Σ is C^4 smooth non-planar and $\mathcal{K} = \int_\Sigma K \,\mathrm{d}\Sigma \le 0$ we have $\inf \sigma(H) < \kappa_1^2$. If Σ is asymptotically flat [leaving out again the precise formulation], the the Dirichlet Laplacian has at least one isolated eigenvalue below κ_1^2.

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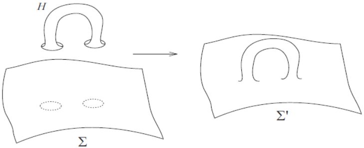
Furthermore, the *Cohn-Vossen inequality* states that

$$\mathcal{K} \le 2\pi\,(2 - 2h - e),$$

where h is the *genus* of Σ and e is the *number of ends*



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Hence $\mathcal{K} < 0$ whenever $h \geq 1$ and we have

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Conclusions of the previous theorem hold whenever Σ is not conformally equivalent to the plane.

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But layers of positive Gauss curvature reveal other interesting property, namely that the spectral properties may depend on the *global geometry* of the region to which the particle is confined.

Example: conical layers

Consider a hard-wall layer of the thickness π built over *conical surface* of an opening angle $\pi - 2\theta$ for some $\theta \in (0, \frac{1}{2}\pi)$,

$$\Sigma_\theta := \{(r, \phi, z) \in \mathbb{R}^3 :\; z = r \sin\theta,\, \phi \in [0, 2\pi)\}$$

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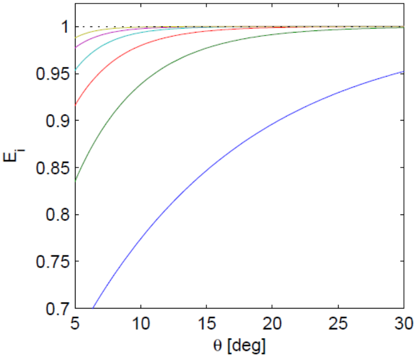
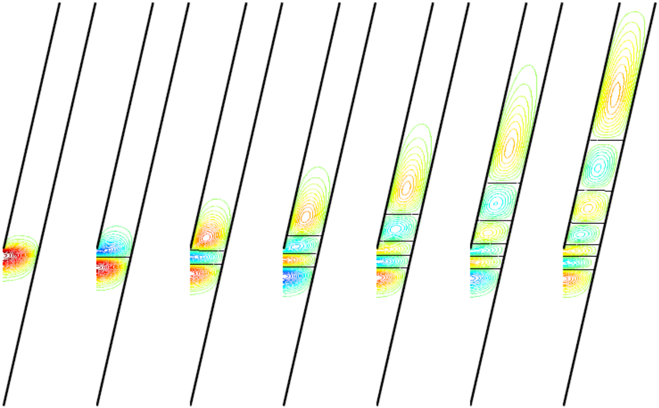
The discrete spectrum infiniteness is related to the fact that the *geodetic circles* on Σ_θ are *shorter* than their counterparts in the plane, which means that the effective attractive potential that behaves asymptotically as $\frac{c}{r^2}$.

Conical layer eigenvalues



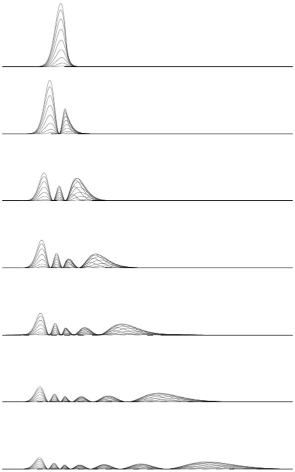
Plot of the dependence of the first six eigenvalues on θ

Conical layer eigenfunctions



Plot of the first seven eigenvalues for $\theta = \frac{5\pi}{36}$

Conical layer probability distributions



Plot of the radial cuts of the first seven probability distributions for $\theta = \frac{5\pi}{36}$

General parabolic layers

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Theorem

We have $\sigma_{\mathrm{ess}}(H) = \left[\left(\frac{\pi}{2a}\right)^2, \infty\right)$ *and* $\#\sigma_{\mathrm{disc}}(H) = \infty$. *Moreover, we have*

$$N_{(\frac{\pi}{2a})^2 - E}(H) \approx \frac{1}{2\pi}\,\frac{\alpha c}{2^\alpha}\,\frac{B\left(\frac{3}{2}, \frac{\alpha}{2} - \frac{1}{2}\right)}{E^{(\alpha-1)/2}} \quad \text{as } E \searrow 0,$$

where $B(\cdot,\cdot)$ *is the Euler beta function, and* $f \approx g$ *means* $f(z), g(z) \to \infty$ *and* $\frac{f(z)}{g(z)} \to 1$ *as* $z \to 0$.

P.E., V. Lotoreichik: Spectral asymptotics of the Dirichlet Laplacian on a generalized parabolic layer, *Int. Eqs Oper. Theory.* **92** (2020), 15

Spiral waveguides

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- spiral shapes appear also in acoustic waveguides

S. Periyannan, P. Rajagopal, K. Balasubramaniam: Multiple temperature sensors embedded in an ultrasonic "spiral-like" waveguide, *AIP Advances* **7** (2017), 035201.

The simplest case: an Archimedean waveguide

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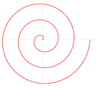

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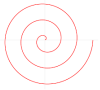

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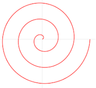

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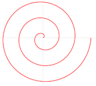

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The quadratic form associated with H_a looks in polar coordinates as

$$q_a :\ q_a[\psi] = \int_0^\infty \int_{r_{\min}(\theta)}^{a\theta} \left[r\left|\frac{\partial\psi}{\partial r}\right|^2 + \frac{1}{r}\left|\frac{\partial\psi}{\partial\theta}\right|^2 \right] \mathrm{d}r\mathrm{d}\theta$$

$$= \int_0^\infty \int_{r/a}^{(r+2\pi a)/a} \left[r\left|\frac{\partial\psi}{\partial r}\right|^2 + \frac{1}{r}\left|\frac{\partial\psi}{\partial\theta}\right|^2 \right] \mathrm{d}\theta\mathrm{d}r$$

defined for all $\psi \in H^1(\Omega_a)$ satisfying Dirichlet condition at points of $\partial\Omega_a$ with $r > 0$ and such that $\lim_{r\to 0+} \frac{\psi(r,\theta)}{\sin\frac{1}{2}\theta}$ exists being independent of θ.

Continuous spectrum of H_a

Theorem

We have $\sigma_{\mathrm{ess}}(H_a) = [(2a)^{-2}, \infty)$. *Furthermore, if I is an open interval away from* $\mathcal{T} = \left\{ \left(\frac{n}{2a}\right)^2 :\ n = 1, 2, \dots \right\}$, *then the spectrum of H_a in I is purely absolutely continuous.*

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$$E_{\tilde{H}_a}(I)[\tilde{H}_a, iA]E_{\tilde{H}_a}(I) \geq -2\,\frac{\partial^2}{\partial r^2}\, E_{\tilde{H}_a}(I) \geq \frac{1}{8}\, E_{\tilde{H}_a}(I)\,;$$

the technical assumptions are satisfied and the bound contains no compact part, hence there are *no embedded eigenvalues* and the spectrum of $\tilde{H}_a$ in the interval I is purely absolutely continuous. $\square$

Discrete spectrum?

The question about the existence of discrete spectrum below $(2a)^{-2}$ is equivalent to the positivity violation of the shifted quadratic form,

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and similarly, $p_{(\alpha,\beta)}[\psi] \geq 0$ for $\beta \leq \pi$. Consequently, the only negative contribution can come from the interval $(\pi, 2\pi)$, in particular, that there can be *at most a finite number of bound states*. However, we are going to argue that the discrete spectrum is in fact *empty*.

Parallel coordinates

Let u be the distance from Γ_a along the inward pointing normal, then the points of $\mathcal{C}_a$ can be parametrized (for $\theta > 2\pi$ at least) as

$$x_1(\theta, u) = a\theta\cos\theta - \frac{u}{\sqrt{1+\theta^2}}\big(\sin\theta + \theta\cos\theta\big),$$
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A natural counterpart to the variable u is the *arc length* of Γ_a given by

$$s(\theta) = a\int_0^\theta \sqrt{1+\xi^2}\,\mathrm{d}\xi = \tfrac{1}{2}a\big(\theta\sqrt{1+\theta^2} + \ln(\theta + \sqrt{1+\theta^2})\big)$$

which for large values of θ behaves as $s(\theta) = \frac{1}{2}a\theta^2 + \mathcal{O}(\ln\theta)$.

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Using it we can express the *curvature* of the spiral as

$$\kappa(\theta) = \frac{2+\theta^2}{a(1+\theta^2)^{3/2}} = \frac{1}{a\theta} + \mathcal{O}(\theta^{-2}) \quad \text{as } \theta\to\infty$$

which means that $\kappa(s) = \frac{1}{\sqrt{2as}} + \mathcal{O}(s^{-1})$ as $s\to\infty$.

What works against curvature-induced bounding?

In the strip $\Sigma_{a,\mathrm{ext}} = \big\{(s,u):\ s > s(2\pi), u \in (0, d(s))\big\}$ we can then pass to a unitarily equivalent operator acting as

$$\hat{H}_{a,\mathrm{ext}}\psi = -\frac{\partial}{\partial s}(1-u\kappa(s))^{-2}\frac{\partial\psi}{\partial s}(s,u) - \frac{\partial^2\psi}{\partial u^2}(s,u) + V(s,u)\psi(s,u),$$

where

$$V(s,u) := -\frac{\kappa(s)^2}{4(1-u\kappa(s))^2} - \frac{u\ddot{\kappa}(s)}{2(1-u\kappa(s))^3} - \frac{5}{4}\,\frac{u^2\dot{\kappa}(s)^2}{(1-u\kappa(s))^4},$$

with an appropriate boundary condition at $s = s(2\pi)$.

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The point is that while the *radial* width of the Archimedes spiral is constant, the 'true', *perpendicular* one, denoted as $d(s)$, is *smaller than $2\pi a$* and only *asymptotically* constant: we have

$$\frac{\pi^2}{d(s)^2} - \frac{1}{4a^2} + V(s,u) = \frac{\pi - u}{2a^2\theta^3} + \mathcal{O}(\theta^{-4}),$$

and as a result, the contributions to the effective potential cancel in the leading order as $\theta \to \infty$ *eliminating thus the curvature-induced attraction*.

A variation: spiral waveguide with a cavity

Let us *'erase' a part of the Dirichlet boundary*, that is, we impose the condition on the 'cut' Archimedean spiral $\Gamma_{a,\beta}$ for some $\beta > 0$, where $\Gamma_{a,\beta} = \{r = a\theta :\, \theta \geq \beta\}$. The particle thus 'lives' in $\mathcal{C}_{a,\beta} := \mathbb{R}^2 \setminus \Gamma_{a,\beta}$ and its Hamiltonian, modulo unimportant physical constants, is

$$H_{a,\beta} = -\Delta_{\mathrm{D}}^{\mathcal{C}_{a,\beta}},$$

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Proposition

There is a critical $\beta_1 = 2j_{0,1} \approx 4.805 \approx 1.531\pi$ *such that* $\sigma_{\mathrm{disc}}(H_{a,\beta}) \neq \emptyset$ *holds for all* $\beta > \beta_1$

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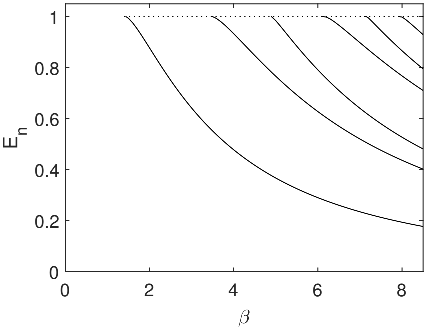
Eigenvalues

One can solve the problem numerically using *FEM technique*; here is how the eigenvalues of $H_{1/2,\beta}$ depend on the angle β:



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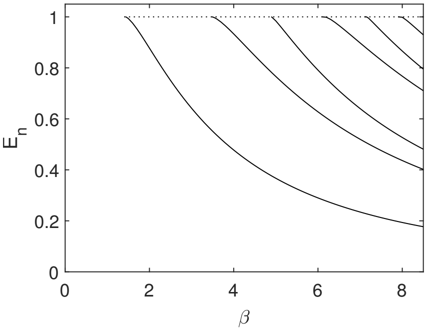

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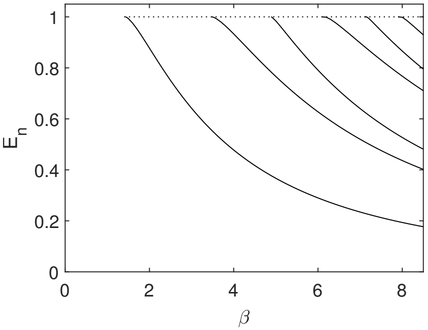

As expected, they are monotonously decreasing functions. We also can identify the critical angle at which the first eigenvalue appears to be $\beta_1 \approx 1.43 \approx 0.455\pi$, a much smaller value than the above sufficient condition; what is more important, it provides the *indication that the discrete spectrum of the 'full' Archimedean spiral region is void*.

Eigenfunctions

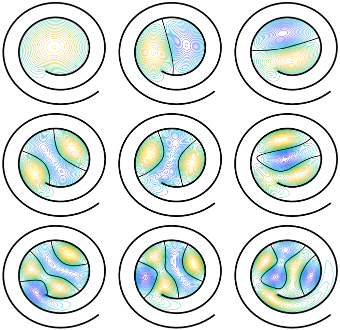

Figure: The first nine eigenfunctions of $H_{1/2,21/2}$ shown through their horizontal levels. The corresponding energies are 0.1280, 0.2969, 0.3456, 0.5312, 05811, 0.6825, 0.8266, 0.8852, and 0.9768, respectively.

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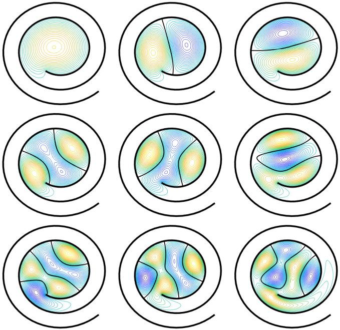

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The results agree with the Courant nodal domain theorem; the nodal lines are situated in the cavity only which, as well the finiteness of the spectrum, corresponds nicely to the observation that the part of $\mathcal{C}_{a,\beta}$ referring to the angles $\theta > \max\{2\pi, \beta_1\}$ is a *classically forbidden zone*.

A variation: multi-arm Archimedean waveguide

Let Γ_a^m be the union of m Archimedean spirals with slope $a > 0$ and an angular shift, $\Gamma_a^m = \{r = a\big(\theta - \frac{2\pi j}{m}\big) :\, \theta \geq \frac{2\pi j}{m},\, j = 0, \dots, m-1\}$. As before we consider its complement $\mathcal{C}_a^m := \mathbb{R}^2 \setminus \Gamma_a^m$ and the operator

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The analysis is similar, but there is a difference coming from regularity of the boundary. For $m = 2$ the set $\mathcal{C}_a^2$ consists of two connected components and has a *smooth boundary*, for $m \geq 3$ it consists of m connected components separated by the branches of Γ_a^m, each of them them has an angle at the origin of coordinates which is $\frac{2\pi}{m}$, that is, *convex*; this means that for $m \geq 2$ the *singular component is missing*.

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It is sufficient to consider one connected component of $\mathcal{C}_a^m$ only, i.e. the operator $\tilde{H}_a^m = -\frac{\partial^2 f}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} - \frac{1}{4r^2}$ referring to the skewed strip

$$\Omega_a^m := \Big\{(r, \theta) :\, r \in (r_{\min}^m(\theta), a\theta), \theta > 0\Big\},$$

where $r_{\min}^m(\theta) := \max\big\{0, a\big(\theta - \frac{2\pi}{m}\big)\big\}$ with $D(H_a^m) = \mathcal{H}^2(\Omega_a^m) \cap \mathcal{H}_0^1(\Omega_a^m)$.

Spectrum of multi-arm spiral region

Proposition

$\sigma(H_a^m) = \left[\left(\frac{m}{2a}\right)^2, \infty\right)$ *for any natural* $m \geq 2$*. The spectrum is absolutely continuous outside* $\mathcal{T}_m = \left\{\left(\frac{mn}{2a}\right)^2 : n = 1, 2, \dots\right\}$ *and its multiplicity is divisible by* m*.*

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Proof sketch: The multiplicity claim is obvious. The above arguments used to determine the essential spectrum and to prove its absolute continuity outside the thresholds modify easily.

Furthermore, *the discrete spectrum is void*. Indeed, since the domain is now 'pure Sobolev', the bottom part, $r = 0$, of the skewed strip supports Dirichlet condition. This means that

$$p^m_{(\alpha,\beta)}[\psi] \geq 0 \qquad \text{now for any } \; 0 \leq \alpha < \beta \leq \infty$$

so that $q_a^m[\psi] - \left(\frac{m}{2a}\right)^2 \|\psi\|^2 \geq p^m_{(0,\infty)}[\psi] \geq 0$ for any $\psi \in \mathrm{dom}[q_a^m]$. $\square$

Eigenfuctions

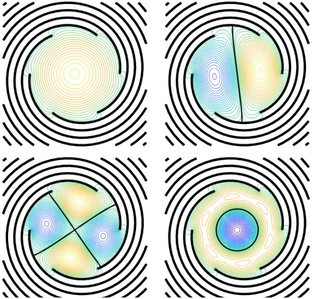
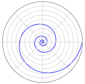


Figure: The jth eigenfunction, $j = 1, 2, 4, 6$, of $H^6_{3,2\pi}$, the corresponding energies are 0.1296, 0.3282, 0.5871, and 0.6783, respectively.

Here we plot result for a six-arm spiral region with the central cavity. As expected, with the growing m the eigenfunctions – with the possible exception of states close to the threshold – become similar to those of the Dirichlet Laplacian in a disc; it is instructive to compare the nodal lines to those of the single arm region shown above.

General spirals

There are many spirals beyond the Archimedean case, for instance



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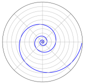

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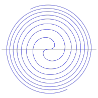

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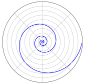

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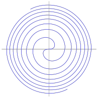

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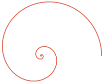

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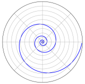

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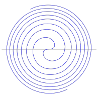

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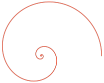

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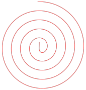

Atzema

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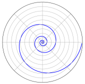

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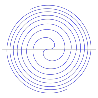

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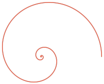
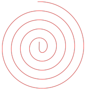
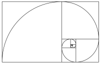


Poinsot



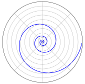
Atzema

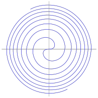


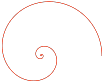
Fibonacci

General spirals

There are many spirals beyond the Archimedean case, for instance


logarithmic


Fermat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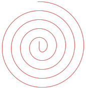
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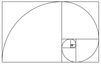
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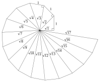
Theodorus

Source: Wikipedia

A spiral curve Γ can be described in polar coordinates as the family of points $(r(\theta), \theta)$, where $r(\cdot)$ is a given increasing function. Let us assume that $r(\cdot)$ is a *C^2-smooth function* excluding thus well-known curves such as *Fibonacci spiral*, *spiral of Theodorus*, etc.

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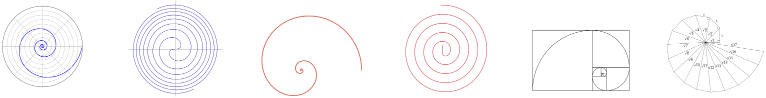

logarithmic Fermat Poinsot Atzema Fibonacci Theodorus

Source: Wikipedia

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Remarks: (i) The spirals considered are semi-infinite $r : \mathbb{R}_+ \to \mathbb{R}_+$. One can also consider *'fully' infinite spirals* for which $r : \mathbb{R} \to \mathbb{R}_+$.

(ii) *A warning:* It matters that the boundary is *Dirichlet*, for *Neumann* the spectral properties can be completely different as the well-known Simon's example, in which $r(\theta) = \frac{3}{4} + \frac{1}{2\pi} \arctan\theta$, shows.

B. Simon: The Neumann Laplacian of a jelly roll, *Proc. AMS* **114** (1992), 783–785.

General spirals

The monotonicity of r means that Γ does not intersect itself, in other words, the *width function* a : $a(\theta) = \frac{1}{2\pi}\big(r(\theta) - r(\theta - 2\pi)\big)$ is positive for any $\theta \geq 2\pi$. The 'inward' coil width is $2\pi a(\theta)$; we make this choice with the correspondence to the Archimedean case in mind).

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Multiarm-arm spirals are similarly described by an m-tuple of increasing functions $r_j: [\theta_j, \infty) \to \mathbb{R}_+,\ j = 0, 1, \dots, m-1$ referring to angles $0 = \theta_0 < \theta_1 < \cdots < \theta_{m-1} < 2\pi$ satisfying

$$a_j(\theta) := \frac{1}{2\pi}\big(r_j(\theta) - r_{j+1}(\theta)\big) > 0 \quad \text{for} \;\; \theta \geq \theta_j$$

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Note that a two-arm spiral can also be alternatively described by means of a function $r: \mathbb{R} \to \mathbb{R}$ such that $\pm r(\theta) > 0$ for $\pm\theta > 0$ if we interpret *negative radii* as describing vectors rotated by π.

Types of general spirals

Asymptotic properties of the width function are decisive. We call a spiral-shaped region $\mathcal{C}$ *simple* if the function $a(\cdot)$ is monotonous (or monotonous on both the halflines $\mathbb{R}_\pm$).

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A spiral-shaped region is called *asymptotically Archimedean* if there is an $a_0 \in \mathbb{R}$ such that $\lim_{\theta\to\infty} a(\theta) = a_0$, for multi-arm spirals this means finite limits of all the a_j.

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A region $\mathcal{C}$ is obviously unbounded *iff* $\lim_{\theta\to\infty} r(\theta) = \infty$. If the limit is finite, $\lim_{\theta\to\infty} r(\theta) = R$, the closure $\overline{\mathcal{C}}$ is contained in the circle of radius R, it may or may not be simply connected as the example of *Simon's jelly roll* mentioned above shows (and the Neumann Laplacian spectrum in this region is *purely continuous*).

Description of general spiral regions

The Hamiltonian domain is $D(H_r) = \mathcal{H}^2(\Omega_r) \cap \mathcal{H}^1_0(\Omega_r) \oplus \mathbb{C}(\psi_{\rm sing})$, with the singular element missing if the boundary is convex around the origin. In polar coordinates H_r is an opeator on a *skewed strip*, now of a generally *nonconstant width*

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$$\begin{aligned} q_r :\ q_r[\psi] &= \int_0^\infty \int_{r_{\min}(\theta)}^{r(\theta)} \Big[r\Big|\frac{\partial\psi}{\partial r}\Big|^2 + \frac{1}{r}\Big|\frac{\partial\psi}{\partial\theta}\Big|^2\Big] \mathrm{d}r\mathrm{d}\theta \\ &= \int_0^\infty \int_{\theta^{-1}(r)}^{\theta^{-1}(r)+2\pi} \Big[r\Big|\frac{\partial\psi}{\partial r}\Big|^2 + \frac{1}{r}\Big|\frac{\partial\psi}{\partial\theta}\Big|^2\Big] \mathrm{d}\theta\mathrm{d}r \end{aligned}$$

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We can also express the *arc length* of Γ and its *curvature*; they are

$$s(\theta) = \int_0^\theta \sqrt{\dot r(\xi)^2 + r(\xi)^2}\,\mathrm{d}\xi \quad \text{and} \quad \kappa(\theta) = \frac{r(\theta)^2 + 2\dot r(\theta)^2 - r(\theta)\ddot r(\theta)}{(r(\theta)^2 + \dot r(\theta)^2)^{3/2}}.$$

Strictly expanding spiral regions

In contrast to the Archimedean case, it may not be possible to amend the arclength with the orthogonal coordinate u to parametrize $\mathcal{C}_r$ by

$$x_1(\theta,u)= r(\theta)\cos\theta - \frac{u}{\sqrt{\dot r(\theta)^2+r(\theta)^2}}\,\big(\dot r(\theta)\sin\theta + r(\theta)\cos\theta\big),$$

$$x_2(\theta,u) = r(\theta)\sin\theta + \frac{u}{\sqrt{\dot r(\theta)^2+r(\theta)^2}}\,\big(\dot r(\theta)\cos\theta - r(\theta)\sin\theta\big).$$

The reason is that for strictly expanding spirals the inward normal at a point may not intersect the previous spiral coil; it is easy to check that in the examples of a *logarithmic spiral*, $r(\theta) = a\,\mathrm{e}^{k\theta}$ with $a, k > 0$, or *hyperbolic spiral*, $r(\theta) = a\theta^{-1}$.

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Fortunately, some properties of H_r can be derived without the use of the locally orthogonal system. Using suitable Weyl sequences one can prove the following claim:

Proposition

$\sigma(H_r) = \sigma_{\mathrm{ess}}(H_r) = [0,\infty)$ *holds if* $\mathcal{C}$ *is simple and strictly expanding.*

Strictly shrinking spiral regions

On the other hand, parallel coordinates *can be used*, possibly outside a compact region, if $\mathcal{C}$ is generated by a shrinking or an asymptotically Archimedean spirals.

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We combine bracketing with the unitarily equivalent form of the operator in parallel coordinates,

$$\hat{H}^{\mathrm{D}}_{\mathrm{nc}} \geq -\frac{\partial}{\partial s}(1-u\kappa(s))^{-2}\frac{\partial}{\partial s} + \frac{\pi^2}{d(s)^2} + V(s,u)$$

and similarly for $\hat{H}^{\mathrm{N}}_{\mathrm{nc}}$. Since $d(s) \to 0$ as $s \to \infty$ holds is a strictly shrinking region, the sum of the two last term explodes in the limit, and in the standard way we can check the following claim:

Proposition

If $\mathcal{C}$ is simple and strictly shrinking, the spectrum of H_r is purely discrete.

Example: Fermat spiral

For *Fermat spiral*, $r(\theta)^2 = b^2\theta$, we have $a(\theta) = \frac{1}{2}b\theta^{-1/2} + \mathcal{O}(\theta^{-3/2})$ so the spectrum is discrete

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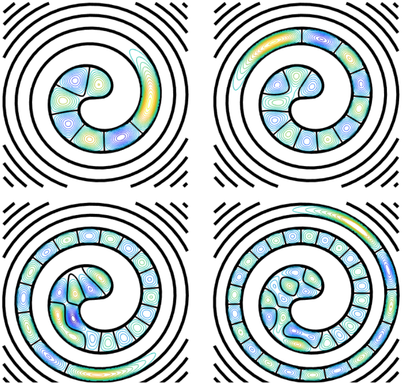

Fermat spiral region: number of eigenvalues

The dominant contribution to the eigenvalue count comes from the transverse confinement potential $v(\theta) = \left(\frac{\pi}{d(\theta)}\right)^2$. For Fermat spiral region this leads to the asymptotics $N(E) \approx \frac{1}{64}\, b^4\, E^2$ as $E \to \infty$.

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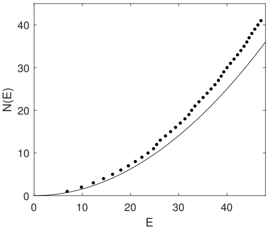

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M. van den Berg, E.B. Davies: Heat flow out of regions in $\mathbb{R}^m$, *Math. Z.* **202** (1989), 463–482.

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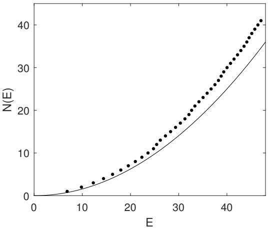

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On the other hand, one can derive a *Lieb-Thirring-type inequality* which shows that *asymptotically* it is only the spiral width here that matters.

D. Barseghyan, P.E.: Spectral estimates for Dirichlet Laplacian on spiral-shaped regions, *J. Spect. Theory*, to appear; `arXiv:2206.14058`

Asymptically Archimedean regions

Between the above discussed extremes the situation is much more interesting. Modifying the argument in the Archimedean case we get

Proposition

If the spiral Γ *is asymptotically Archimedean with* $\lim_{\theta\to\infty} a(\theta) = a_0$, *we have* $\sigma_{\rm ess}(H_r) = [(2a_0)^{-2}, \infty)$*. In the case of a multi-arm region with* $\lim_{\theta\to\infty} a_j(\theta) = a_{0,j}$, *the essential spectrum is* $[(2\overline{a})^{-2}, \infty)$, *where* $\overline{a} := \max_{0\le j\le m-1} a_{0,j}$.

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The question about the discrete spectrum is more subtle and the type of asymptotics is decisive. Let us consider the spiral

$$r(\theta) = a_0\theta + b_0 - \rho(\theta),$$

where $\rho(\cdot)$ is a positive function such that, $\lim_{\theta\to\infty} \rho(\theta) = 0$; for the sake of definiteness we restrict our attention to functions satisfying

$$\dot\rho(\theta) = -\frac{c}{\theta^\gamma} + \mathcal{O}(\theta^{-\gamma-1}) \quad \text{as} \quad \theta\to\infty \quad \text{with} \quad 1<\gamma<3.$$

Infinite discrete spectrum

Theorem

For the described $r(\cdot)$, $\#\sigma_{\mathrm{disc}}(H_r) = \infty$ holds for any $c > 0$.

P.E., M. Tater: Spectral properties of spiral-shaped quantum waveguides, *J. Phys. A: Math. Theor.* **53** (2020), 505303

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Proof sketch: By a variational argument, using the trial functions $\psi_{0,\lambda}$ with mollifier μ that gave the essential spectrum in the Archimedean case. After a straightforward computation we get for the shifted quadratic form

$$p[\psi_{0,\lambda}] < \lambda \frac{4\pi}{a_0} \|\dot{\mu}\|^2 - \Big(\frac{4\pi^2 c}{a_0^4} \Big(\frac{a_0}{4} \Big)^{\gamma/2} \lambda^{(\gamma-2)/2} + \mathcal{O}\big(\lambda^{(\gamma'-2)/2}\big) \Big) \|\mu\|^2,$$

where the right-hand side is negative for the scalling parameter λ small enough. Moreover, since the support of μ is compact, one can choose a sequence $\{\lambda_n\}$ such that $\lambda_n \to 0$ as $n \to \infty$ and the *supports of ψ_{0,λ_n} are mutually disjoint* which means that the discrete spectrum of H_r is infinite, accumulating at the threshold $(2a_0)^{-2}$. $\square$

Fermat meets Archimedes

As an example, consider an interpolation between Fermat and Archimedean spirals, in the simplest case described parametrically as

$$r(\theta) = a\sqrt{\theta\big(\theta + \tfrac{b^2}{a^2}\big)}, \quad a, b > 0,$$

with the asymptotic behavior

$$\begin{aligned} r(\theta) &= b\sqrt{\theta} + \frac{a^2}{2b}\theta^{3/2} + \mathcal{O}(\theta^{5/2}), \\ r(\theta) &= a\theta + \frac{b^2}{2a} + \mathcal{O}(\theta^{-1}) \end{aligned}$$

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for $\theta \to 0+$ and $\theta \to \infty$, respectively.

The Fermat spiral is conventionally considered as a *two-arm one* dividing the plane into a pair of mutually homothetic regions, hence we interpolate with the two-arm Archimedean spiral; the essential spectrum is $[a^{-2}, \infty)$.

Fermat meets Archimedes, continued

As for the discrete spectrum, taking the expansion of $r(\theta)$ two terms further, we get $b_0 = \frac{b^2}{2a}$ and

$$\rho(\theta) = \frac{b^4}{8a^3\theta} - \frac{3b^6}{16a^5\theta^2} + \mathcal{O}(\theta^{-3}).$$

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This means that the assumptions of the last proposition hold with with $c = \frac{b^4}{8a^3} > 0$ and $\gamma = 2$, and the the operator H_r has an *infinite discrete spectrum* in $(0, a^{-2})$ accumulating at the threshold.

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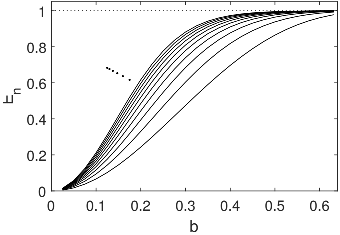
This means that the assumptions of the last proposition hold with with $c = \frac{b^4}{8a^3} > 0$ and $\gamma = 2$, and the the operator H_r has an *infinite discrete spectrum* in $(0, a^{-2})$ accumulating at the threshold.

One can also specify the accumulation rate: the one-dimensional effective potential is in this case $\frac{\pi b^4}{16a^5}\, s^{-1} + \mathcal{O}(s^{-3/2})$, with the leading term of Coulomb type, which shows that the number of eigenvalues *below* $a^{-2} - E$ behaves as

$$\mathcal{N}_{a^{-2}-E}(H_r) = \frac{\pi b^4}{32a^5}\, \frac{1}{\sqrt{E}} + o(E^{-1/2}) \quad \text{if} \quad E \to 0+$$

Eigenvalues

Let us plat the lowest eigenvalues of such a region as functions of b:



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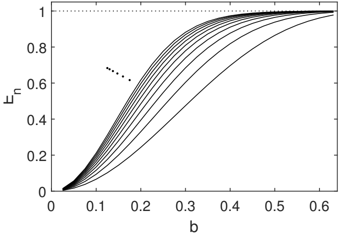

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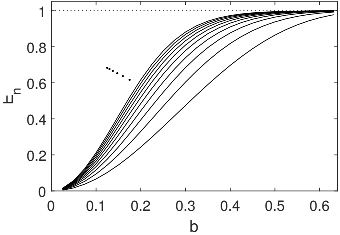

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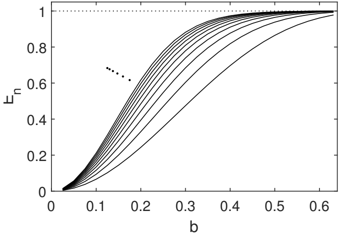

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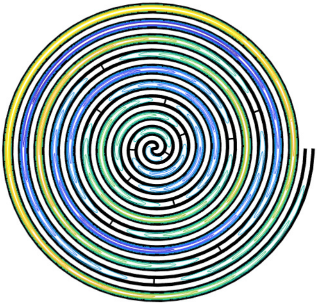
Remark: Experimentalists often label their spirals as Archimedean, but in fact *they are not*, being produced by coiling fibers of a *fixed cross section*, hence their transverse width is *constant* with respect to θ, in contrast to the true Archimedean spiral. Such waveguides behave asymptotically rather as the current interpolation with $\frac{b}{a} = (2\pi)^{-1/4} \approx 0.632$.

An eigenfunction example

Here is the eigenfunction with $E_{14} = 0.999952$ referring to $b = (2\pi)^{-1/4}$.



The difference from the two-arm Archimedean region is *hardly perceptible by a naked eye*, however, the discrete spectrum is now not only non-void but it is rich with the eigenfunctions the tails of which have a distinctively *quasi-one-dimensional character*.

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- The existence of bound states in spiral waveguides depends on the asymptotic behavior of their width; the problem is subtle in the *asymptotically Archimedean* case.
- In contrast to mathematics, *'physicist's Archimedean waveguides'* have numerous weakly bound states.