Guided quantum dynamics

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With thanks to all my collaborators

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- Concerning the terminology: at the beginning one spoke about *QM on graphs*. The term *quantum graph* was coined by Uzy Smilansky at the end of the 90s, and he later expressed regrets about that.

Hall effect

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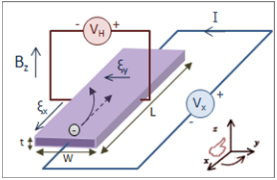

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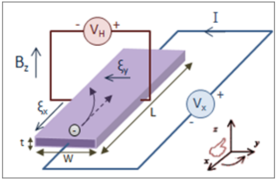

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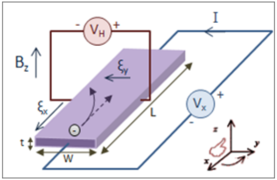

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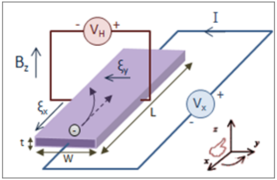

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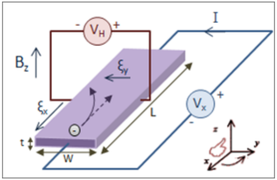

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In contrast to the 'usual' quantum Hall effect, its mechanism is not well understood; it is conjectured that it comes from *internal magnetization* in combination with the *spin-orbit interaction*.

Modeling anomalous Hall effect

Recently a *quantum-graph model* of the AHE was proposed in which the material structure of the sample is described by lattice of *δ-coupled rings* (topologically equivalent to the *square lattice* we will discuss later)

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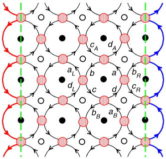

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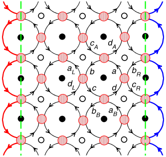

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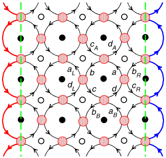

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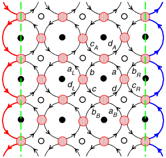

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Looking at the picture we recognize a *flaw in the model*: to mimick the rotational motion of atomic orbitals responsible for the magnetization, the authors had to impose 'by hand' the requirement that the electrons move only one way on the loops of the lattice. Naturally, such an assumption *cannot be justified from the first principles!*

Breaking the time-reversal invariance

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$$U = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

Writing the coupling componentwise for vertex of degree N, we have

$$(\psi_{j+1}-\psi_j) + i(\psi'_{j+1}+\psi'_j) = 0\,, \quad j \in \mathbb{Z} \pmod N\,,$$

which is non-trivial for $N \geq 3$ and obviously non-invariant w.r.t. the reverse in the edge numbering order

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Star graphs: spectrum and scattering

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with m running through $1, \dots, [\frac{N}{2}]$ for N odd and $1, \dots, [\frac{N-1}{2}]$ for N even. Thus $\sigma_{\mathrm{disc}}(H)$ is *always nonempty*

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However, caution is needed; the formal limits lead to a *false result* if $+1$ or -1 are eigenvalues of U. A *counterexample* is the (scale invariant) Kirchhoff coupling where U has only ± 1 as its eigenvalues; the on-shell S-matrix is then independent of k and it is *not* a multiple of the identity.

The vertex parity enters the game

Denoting for simplicity $\eta := \frac{1-k}{1+k}$, a straightforward computation gives

$$S_{ij}(k) = \frac{1-\eta^2}{1-\eta^N} \left\{ -\eta \, \frac{1-\eta^{N-2}}{1-\eta^2} \, \delta_{ij} + (1-\delta_{ij}) \, \eta^{(j-i-1) (\mathrm{mod}\, N)} \right\},$$

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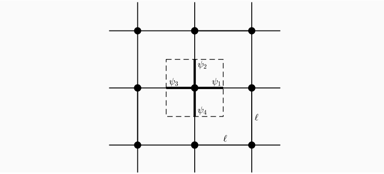
Let us look how this fact influences spectra of *periodic* quantum graphs.

Comparison of two lattices



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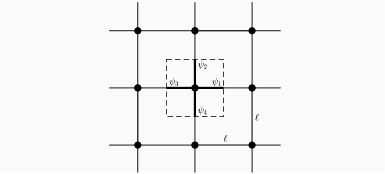

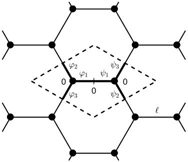

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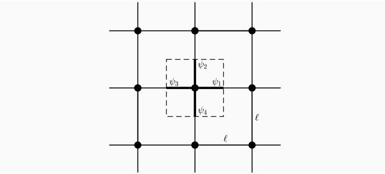

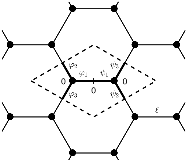

Spectral condition for the two cases are easy to derive,

$$16i\, \mathrm{e}^{i(\theta_1+\theta_2)}\, k\, \sin k\ell \big[(k^2-1)(\cos\theta_1+\cos\theta_2)+2(k^2+1)\cos k\ell\big]=0$$

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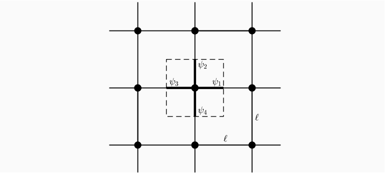

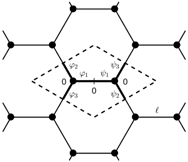

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and respectively

$$16i\, \mathrm{e}^{-i(\theta_1+\theta_2)}\, k^2 \sin k\ell \left(3+6k^2-k^4+4d_\theta(k^2-1)+(k^2+3)^2\cos 2k\ell\right)=0\,,$$

where $d_\theta := \cos\theta_1+\cos(\theta_1-\theta_2)+\cos\theta_2$ and $\frac{1}{\ell}(\theta_1,\theta_2)\in[-\frac{\pi}{\ell},\frac{\pi}{\ell}]^2$ is the quasimomentum

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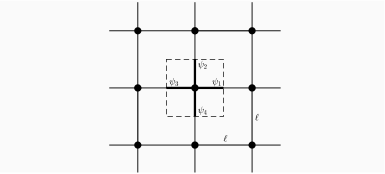

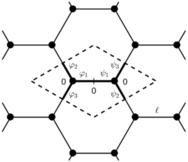

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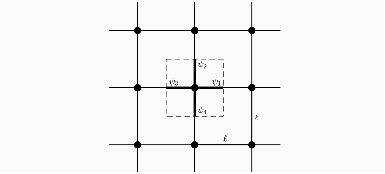

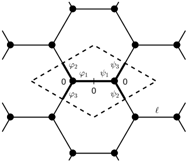

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P.E., M. Tater: Quantum graphs with vertices of a preferred orientation, *Phys. Lett.* **A382** (2018), 283–287.

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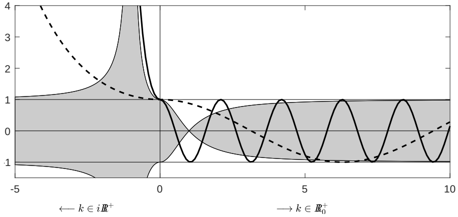

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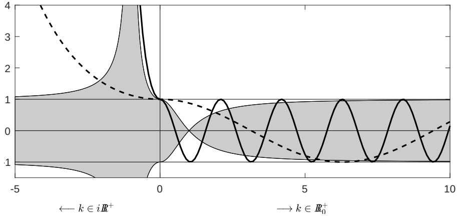

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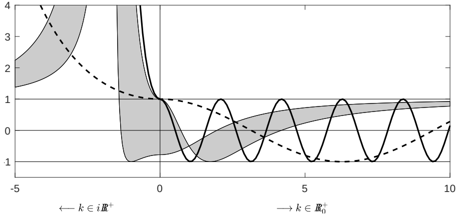

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Let us mention one more involved choice of the vertex coupling.

An interpolation

One can *interpolate* between the δ-coupling and the present one taking e.g., for U the *circulant matrix* with the eigenvalues

$$\lambda_k(t) = \begin{cases} \mathrm{e}^{-i(1-t)\gamma} & \text{for } k = 0; \\ -\,\mathrm{e}^{i\pi t\left(\frac{2k}{n}-1\right)} & \text{for } k \geq 1 \end{cases}$$

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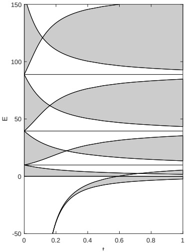

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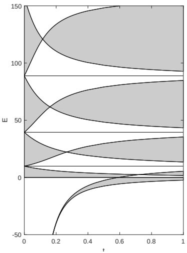

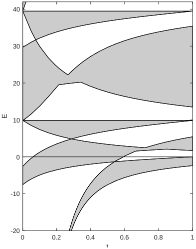

P.E., O. Turek, M. Tater: A family of quantum graph vertex couplings interpolating between different symmetries, *J. Phys. A: Math. Theor.* **51** (2018), 285301.

Another topic: band edges positions

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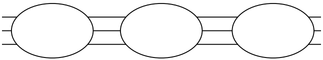

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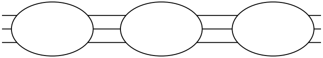

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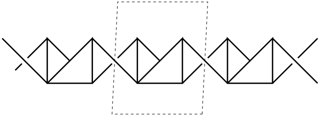

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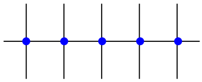

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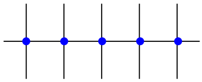

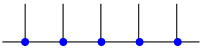

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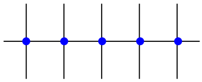

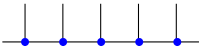

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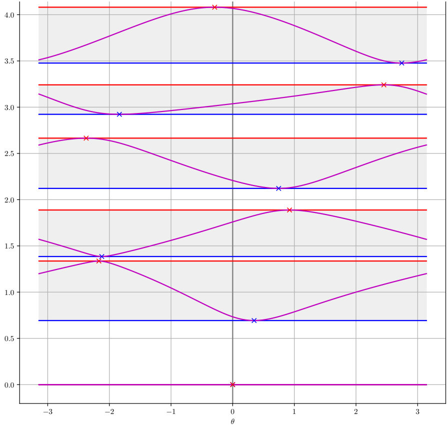
- and what about the dispersion curves?

Two-sided comb: dispersion curves



P.E., Daniel Vašata: Spectral properties of $\mathbb{Z}$ periodic quantum chains without time reversal invariance, *in preparation*

Discrete symmetry: Platonic solid graphs

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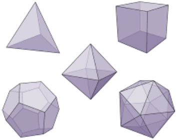

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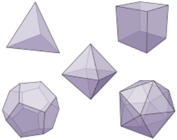

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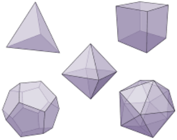

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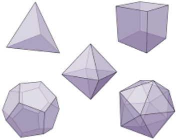

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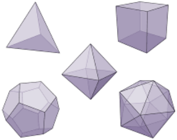

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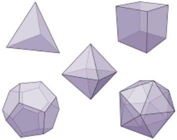

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P.E., J. Lipovský: Spectral asymptotics of the Laplacian on Platonic solids graphs, *J. Math. Phys.* **60** (2019), 122101

Another periodic graph model

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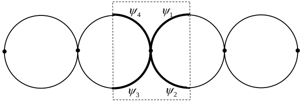
Let us look what this coupling influences graphs *periodic in one direction*. Consider again a *loop chain*, first *tightly connected*



The spectrum of the corresponding Hamiltonian looks as follows:

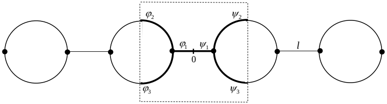
Theorem

The spectrum of H_0 consists of the absolutely continuous part which coincides with the interval $[0, \infty)$, and a family of infinitely degenerate eigenvalues, the isolated one equal to -1, and the embedded ones equal to the positive integers.

M. Baradaran, P.E., M. Tater: Ring chains with vertex coupling of a preferred orientation, *Rev. Math. Phys.* **33** (2021), 2060005.

A loosely connected chain

Replace the direct coupling of adjacent rings by connecting segments of length $\ell > 0$, still with the same vertex coupling.



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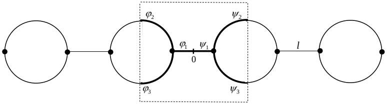

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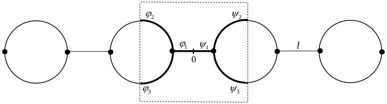

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- *Away of the non-negative integers the spectrum is absolutely continuous having a band-and-gap structure.*

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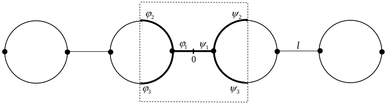

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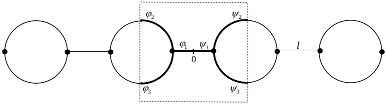
The spectrum of H_ℓ has for any fixed $\ell > 0$ the following properties:

- *Any non-negative integer is an eigenvalue of infinite multiplicity.*
- *Away of the non-negative integers the spectrum is absolutely continuous having a band-and-gap structure.*
- *The negative spectrum is contained in $(-\infty, -1)$ consisting of a single band if $\ell = \pi$, otherwise there is a pair of bands and $-3 \notin \sigma(H_\ell)$.*

A loosely connected chain

Replace the direct coupling of adjacent rings by connecting segments of length $\ell > 0$, still with the same vertex coupling.



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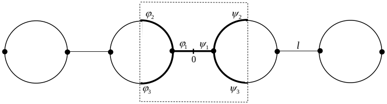

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- *The positive spectrum has infinitely many gaps.*
- $P_\sigma(H_\ell) := \lim_{K\to\infty} \frac{1}{K} |\sigma(H_\ell) \cap [0, K]| = 0$ *holds for any $\ell > 0$.*

The limit $\ell \to 0+$

The quantity $P_\sigma(H_\ell)$ in the last claim of the theorem is the *probability of being in the spectrum*, which was introduced in

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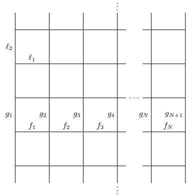
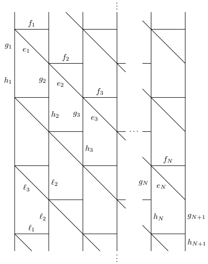
Note also that if we violate the mirror symmetry of the chain, we have instead $P_\sigma(H_0) = \frac{1}{2}$ independently of where exactly we place the vertex (irrationality not needed due the very simple form of spectral condition).

M. Baradaran, P.E., M. Tater: Spectrum of periodic chain graphs with time-reversal non-invariant vertex coupling, *Ann. Phys.* **443** (2022), 168992

One more example: transport properties

Consider strips cut of the following two types of lattices:





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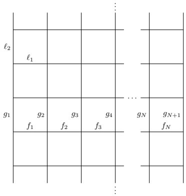

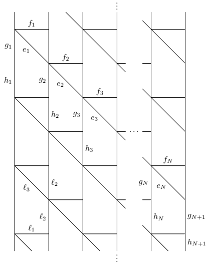

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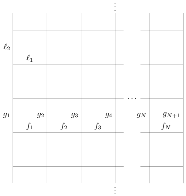

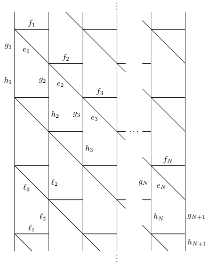

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This time we ask in which part of the ‘guide’ are the generalized eigenfunction *dominantly supported*

Transport properties, continued

Theorem

- *In the rectangular-lattice strip, for a fixed* $K \in \left(0, \frac{1}{2}\pi\right)$*, consider* $k > 0$ *obeying* $k \notin \bigcup_{n\in\mathbb{N}_0} \left(\frac{n\pi-K}{\ell_2}, \frac{n\pi+K}{\ell_2}\right)$*. With the natural normalization of the generalized eigenfunction corresponding to energy* k^2*, its components at the leftmost and rightmost vertical edges are of order* $\mathcal{O}(k^{-1})$ *as* $k \to \infty$*.*

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Adopting the same normalization as above and denoting by $q_j^{(m)}$ *with* $m = 1, \dots, 8$, *the coefficients of wave function components for the edges directed down and right from vertices of the* j*th vertical line, we have* $q_j^{(m)} = \mathcal{O}(k^{1-j})$ *as* $k \to \infty$.

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Remark: Note that the 'brick-lattice' strip is *not* a topological insulator!

$\mathcal{PT}$-symmetry

Having two research areas, each based of a strong concept, it is natural to look for connecting links. This applies, in particular, to quantum graphs and *$\mathcal{PT}$-symmetry*, also intensely studied in the last three decades.

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The focus is, of course, on *nontrivial situations* when neither parity nor the time-reversal invariance were preserved but their composition was. The known examples of $\mathcal{PT}$-symmetry in quantum graphs go beyond the class of self-adjoint Hamiltonians.

A. Hussein, D. Krejčiřík, P. Siegl: Non-selfadjoint quantum graphs, *Trans. Amer. Math. Soc.* **367** (2015), 2921–2957.

P. Kurasov, B. Majidzadeh Garjani: Quantum graphs: $\mathcal{PT}$-symmetry and reflection symmetry of the spectrum, *J. Math. Phys.* **58** (2017), 023506.

D.U. Matrasulov, K.K.Sabirov, J.R. Yusupov: $\mathcal{PT}$-symmetric quantum graphs, *J. Phys. A: Math. Theor.* **52** (2019), 155302.

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As usual in QM, a symmetry is described by an operator $\mathcal{H} \to \mathcal{H}$ leaving the Hamiltonian is invariant. In our case the nontrivial part concerns the matching condition: a particular symmetry is associated with an invertible map in the space of the boundary values, $\Theta : \mathbb{C}^n \to \mathbb{C}^n$, such that we have $(U-I)\Theta\Psi(0) + i(U+I)\Theta\Psi'(0) = 0$ for all admissible Ψ, or equivalently

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One asks which operators are associated with the parity and time reversal transformations. The latter is simpler. Operator $\Theta_{\mathcal{T}}$ is *antilinear* and *idempotent*, in the absence of internal degrees of freedom it is just the *complex conjugation*. Using the unitarity, $U^T \bar{U} = \bar{U} U^T = I$ we see that $\bar{\Psi}$ satisfies the matching condition with the *transposed matrix*, that is,

$$\Theta_{\mathcal{T}}^{-1} U \Theta_{\mathcal{T}} = \Theta_{\mathcal{T}} U \Theta_{\mathcal{T}} = U^T,$$

and consequently, the *H_U is $\mathcal{T}$-invariant if and only if $U = U^T$*.

How to describe mirror transformations?

This also immediately implies that a (self-adjoint) quantum graph *is $\mathcal{PT}$-symmetric if and only if the mirror transformation acts analogously,*

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To see which operator can facilitate the similarity between U and U^T, we use the *unitarity* of the matrix: there is a unitary V such that VUV^* is *diagonal*, and as such equal to its transpose. It follows that the matrix Θ satisfying $\Theta U \Theta = U^T$ is of the form $\Theta = V^T V$.

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We know how V looks like: the jth column of V^* coincides with ϕ_j^T, where ϕ_j is the jth normalized eigenvector of U. Consequently, we have

$$\Theta_{ij} = (\bar{\phi}_i, \phi_j), \quad i,j = 1,\dots,n\,;$$

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The situation changes, however, when we restrict our attention to the subset of *circulant* matrices, i.e. those of the form

$$U = \begin{pmatrix} c_1 & c_2 & \cdots & c_{n-1} & c_n \\ c_n & c_1 & c_2 & & c_{n-1} \\ \vdots & c_n & c_1 & \ddots & \vdots \\ c_3 & & \ddots & \ddots & c_2 \\ c_2 & c_3 & \cdots & c_n & c_1 \end{pmatrix}.$$

Circulant matrices

The unitarity requires that

$$c_j = \frac{1}{n}\left(\lambda_1 + \lambda_2\omega^{-j} + \lambda_3\omega^{-2j} + \cdots + \lambda_n\omega^{-(n-1)j}\right), \quad j = 1, \ldots, n,$$

where λ_j, $j = 1, \ldots, n$, are eigenvalues of U and $\omega := \mathrm{e}^{2\pi i/n}$. The corresponding eigenvectors are independent of the choice of the c_j's,

$$\phi_j = \frac{1}{\sqrt{n}}\left(1, \omega^j, \omega^{2j}, \ldots, \omega^{(n-1)j}\right)^T, \quad j = 1, \ldots, n.$$

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Furthermore, the eigenvalues can be written in terms of the matrix entries as $\lambda_j = \sum_{k=1}^n c_k\omega^{j(k-1)}$. The diagonalization is achieved in this case by the *discrete Fourier transformation*,

$$V^* = \frac{1}{\sqrt{n}}\begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \dots & \omega^{(n-1)} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \dots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \dots & \omega^{(n-1)^2} \end{pmatrix}.$$

Mirror transformation for circulant matrices

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This has the needed properties, preserving the edge e_1, as well as e_{k+1} if $n = 2k$, and among the remaining ones *it switches e_j with e_{n+2-j}*, and moreover, the same will be true if we renumber the edges.

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The examples we discussed above belong, of course, to the latter subset.

A purely Robin coupling

To elucidate further the role played by the *absence of the Dirichlet component* in the vertex coupling, consider another interpolation: the coupling with

$$U = \epsilon R, \quad \epsilon = \mathrm{e}^{i\mu}, \quad \mu \in \left(0, \tfrac{2\pi}{n}\right)$$

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In components the matching condition in this case reads

$$\epsilon\psi_{j+1} - \psi_j + i\ell(\epsilon\psi'_{j+1} + \psi'_j) = 0 \pmod{n}$$

and its $\mathcal{PT}$-symmetry is obvious. Putting $\eta := \frac{1-k\ell}{1+k\ell}$ we find

$$S_{ij}(k) = \frac{1}{1-\epsilon^n\eta^n}\Big(-\eta(1-\epsilon^n\eta^{n-2})\delta_{ij} + (1-\delta_{ij})(1-\eta^2)\epsilon(\epsilon\eta)^{(j-i-1)(\mathrm{mod}\, n)}\Big).$$

We have now $\lim_{k\to\infty} S(k) = I$ because of the factor $1-\eta^2$ which cancels out with the prefactor only if $\epsilon = 1$.

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To see how the spectrum changes, let us revisit the square lattice example.

P.E., M. Tater: Quantum graphs: self-adjoint, and yet exhibiting a nontrivial $\mathcal{PT}$-symmetry, *Phys. Lett.* **A416** (2021), 127669

Square lattice example revisited

In the elementary cell of the lattice, we use again the Ansatz

$$\psi_1(x) = a_1 \mathrm{e}^{ikx} + b_1 \mathrm{e}^{-ikx}, \psi_2(x) = a_2 \mathrm{e}^{ikx} + b_2 \mathrm{e}^{-ikx},$$

$$\psi_3(x) = \omega_1 \left(a_1 \mathrm{e}^{ik(x+\ell)} + b_1 \mathrm{e}^{-ik(x+\ell)} \right), \psi_4(x) = \omega_2 \left(a_2 \mathrm{e}^{ik(x+\ell)} + b_2 \mathrm{e}^{-ik(x+\ell)} \right).$$

Using the mentioned matching condition and Floquet at the 'loose' ends, we get a linear system which is solvable if the determinant

$$D \equiv D(\eta, \omega_1, \omega_2) = \begin{vmatrix} -1 & -\eta & \epsilon\eta & \epsilon \\ \epsilon\omega_1\xi^2 & \epsilon\omega_1\bar{\xi}^2\eta & -1 & -\eta \\ -\omega_1\xi^2\eta & -\omega_1\bar{\xi}^2 & \epsilon\omega_2\xi^2 & \epsilon\omega_2\bar{\xi}^2\eta \\ \epsilon\eta & \epsilon & -\omega_2\xi^2\eta & -\omega_2\bar{\xi}^2 \end{vmatrix},$$

where $\omega_j = \mathrm{e}^{i\theta_j}$, $\xi = \mathrm{e}^{ik\ell}$ and $\epsilon = \mathrm{e}^{i\mu}$ with $\mu \in \left(0, \frac{1}{2}\pi\right)$ vanishes

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where $\omega_j = \mathrm{e}^{i\theta_j}$, $\xi = \mathrm{e}^{ik\ell}$ and $\epsilon = \mathrm{e}^{i\mu}$ with $\mu \in \left(0, \frac{1}{2}\pi\right)$ vanishes. This gives

$$8i\epsilon^2 \, \frac{\mathrm{e}^{i(\theta_1+\theta_2)}}{(k+1)^4} \sum_{j=0}^{4} c_j k^j = 0,$$

where

$$c_0 = c_4 = -\sin 2\mu \, \sin^2 k\ell, \quad c_2 = \sin 2\mu \, (1 + 3\cos 2k\ell),$$

$$c_1 = 2\big(2\cos 2\mu \, \cos k\ell - \cos\theta_1 - \cos\theta_2\big) \sin k\ell,$$

$$c_3 = 2\big(2\cos 2\mu \, \cos k\ell + \cos\theta_1 + \cos\theta_2\big) \sin k\ell;$$

for the negative spectrum one has to set $k = i\kappa$ with $\kappa > 0$.

Spectral properties

If $\mu = 0$ the even c_j's are zero and we get the solution discussed above, in particular, the positive spectrum is *dominated by bands* growing linearly.

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$$\cot 2\mu = \frac{-1 - 3\cos 2\ell + 2\sin^2 \ell}{4 \sin 2\ell} = -\cot 2\ell,$$

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We lack now the nice graphical solution we had for $\mu = 0$, but it is not difficult to determine the high-energy asymptotic behavior. Since $c_4 \neq 0$ for $k \neq \frac{\pi n}{\ell}$, spectral bands may exist only *in the vicinity of* $\left(\frac{\pi n}{\ell}\right)^2$, while these point themselves do not belong to $\sigma(H_U)$.

Spectral properties, continued

Furthermore, the width of the nth band on the energy scale is *for a fixed* $\mu \in \left(0, \frac{\pi}{2}\right)$ *asymptotically constant*,

$$\Delta_n \lesssim \frac{8}{\ell} \cot \mu.$$

We stress the fixed value of μ. The band width is not monotonous over the whole interval $\left[0, \frac{\pi}{2}\right]$; as we are approaching the right endpoint, it starts growing again, because $U = iR$ too *has -1 as its eigenvalue*.

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This non-uniform character is also manifested by the fact that we have

$$P_\sigma(H_U) := \lim_{K\to\infty} \frac{1}{K} \left|\sigma(H_U) \cap [0, K]\right| = 0, \quad \mu \in \left(0, \tfrac{\pi}{2}\right),$$

while for both the real-valued $U = R$ and the purely imaginary $U = iR$ the probability is equal to one.

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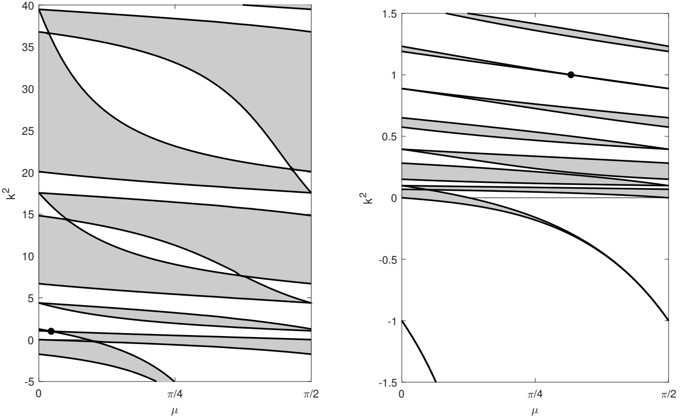
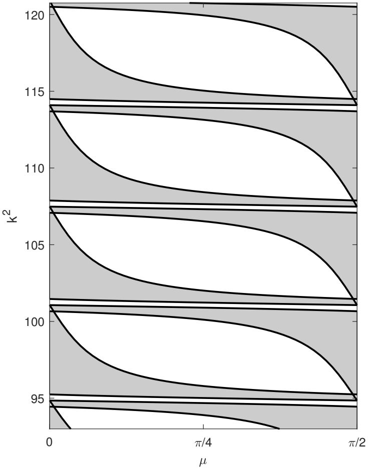
The negative spectrum of H_U has *two bands*. In particular, for large ℓ they are narrow and to shrink to the star-graph eigenvalues referring to $\kappa = \tan \frac{\mu}{2}$ and $\tan\left(\frac{\mu}{2} + \frac{\pi}{4}\right)$ as $\kappa \to \infty$.

The spectrum as a function of μ



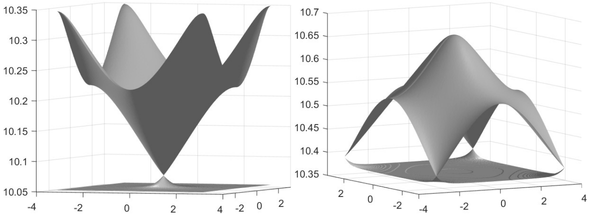
The spectrum of H_U for $\ell = \frac{3}{2}$ and $\ell = 10$, respectively. The dot indicates the flat band at $k = 1$.

High-energy spectrum



The spectrum for $\ell = 10$ again: for a fixed $\mu \in \left(0, \frac{\pi}{2}\right)$ the positive spectral bands get narrower as the energy grows, while at the endpoints of the interval they dominate the spectrum.

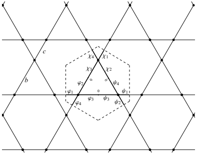
Closing gaps



In distinction to the flat band at $(\mu, k) = \left(\frac{\pi}{2} - \ell, 1\right)$ we have *true band crossings* occurring either in the center of the Brillouin zone or its corners. Here we have dispersion surfaces (in the momentum variable) for $\ell = 10$ at the points of closing gaps, left at $(\mu, k) = (1.55068665, 10.07328547)$, right at $(\mu, k) = (1.55190524, 10.38681556)$. The picture clearly shows the *Dirac cones* at the touching points.

Returning to the original vertex coupling

One can analyze other examples such as *Kagome lattice* with the coupling $U = R$ and its degenerate case, the *triangular lattice*



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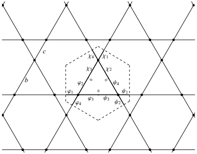

The spectrum – here for $d = b+c = 6$ as a function of c – has a complicated structure with 'true' and flat bands and band-edge crossings

M. Baradaran, P.E.: Kagome network with vertex coupling of a preferred orientation, *J. Math. Phys.* **63** (2022), 083502

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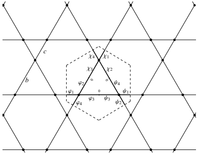

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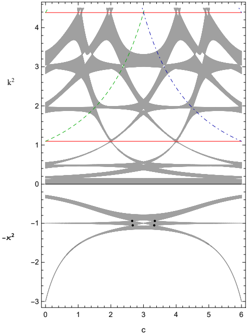

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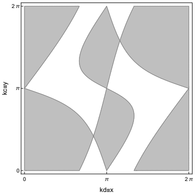
To understand the reason, let us see how the spectral condition looks like,

$$0 \le \frac{5}{4} + \frac{\cos kd \; \cos\frac{k(2c-d)}{2} + \cos\frac{3kd}{2}}{\cos\frac{k(2c-d)}{2} + \cos\frac{kd}{2}} \le \frac{9}{4},$$

hence in the *ergodic situation* we have just to calculate the area of the appropriate part of the torus, in contrast to commensurate edge situations:



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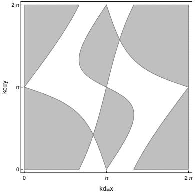

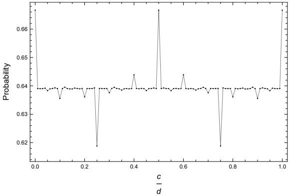

Spectral optimization

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To give one more example, let us mention the *Payne-Pólya-Weinberger inequality*: in the same situation the *ratio* of the first two eigenvalues, $\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)}$, is sharply *maximized* by a ball.

M.S. Ashbaugh, R.D. Benguria: A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions, *Ann. Math.* **135** (1992), 601–628.

Non-simply connected regions

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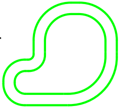
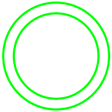
ground state of  $<$ ground state of 

whenever the strip is not a circular annulus.

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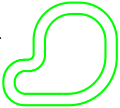
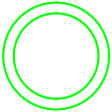
ground state of  $<$ ground state of 

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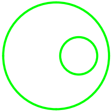
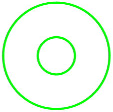
Similarly, for a *circular obstacle in circular cavity* we have

ground state of  $<$ ground state of 

whenever the obstacle is off center; the minimum is reached when it is touching the boundary.

E.M. Harrell, P. Kröger, K. Kurata: On the placement of an obstacle or a well so as to optimize the fundamental eigenvalue, *SIAM J. Math. Anal.* **33** (2001), 240–259.

A leaky loop analogue

Let Γ be a *loop* in $\mathbb{R}^d$, $d \geq 2$, parametrized by its arc length, i.e. a *piecewise differentiable* function $\Gamma : [0, L] \to \mathbb{R}^d$ such that $\Gamma(0) = \Gamma(L)$ and $|\dot{\Gamma}(s)| = 1$ for all but finitely many $s \in [0, L]$

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Theorem

Let $d = 2$. For any $\alpha > 0$ and $L > 0$ we have $\lambda_1(\alpha, \Gamma) \leq \lambda_1(\alpha, \mathcal{C})$, where $\mathcal{C}$ is a circle of perimeter L, the inequality being sharp unless Γ is congruent with $\mathcal{C}$.

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One more time, we employ the generalized *Birman-Schwinger principle* by which there is one-to-one correspondence between eigenvalues $-\kappa^2$ of $H_{\alpha,\Gamma}$ and solutions to the integral-operator equation

$$\mathcal{R}^{\kappa}_{\alpha,\Gamma}\phi = \phi\,, \quad \text{where } \mathcal{R}^{\kappa}_{\alpha,\Gamma}(s,s') := \frac{\alpha}{2\pi} K_0(\kappa|\Gamma(s) - \Gamma(s')|)$$

on $L^2([0, L])$, where K_0 is the Macdonald function.

Rephrasing it as a geometric problem

We employ *inequalities on mean values of chords* denoted as $C_L^p(u)$:

$$\int_0^L |\Gamma(s+u)-\Gamma(s)|^p \mathrm{d}s \le \frac{L^{1+p}}{\pi^p} \sin^p \frac{\pi u}{L}\,, \quad p>0,\ u\in(0,\tfrac{1}{2}L]$$

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Remark: The (reverse) inequalities hold also for $p\in[-2,0)$ showing, e.g., that a *charged loop in the absence of gravity takes a circular form*.

A discrete analogue: polymer loops

Consider the same loop as above with *point interactions* placed at the *arc distances* $\frac{jL}{N}$, $j = 0, \ldots, N_1$, in other words, the formal Hamiltonian

$$H_{\alpha,\Gamma}^{N} = -\Delta + \tilde{\alpha} \sum_{j=0}^{N-1} \delta\Big(x - \Gamma\Big(\frac{jL}{N}\Big)\Big)$$

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Introduce the *generalized boundary values* as the coefficients in the expansion of H_Y^* where H_Y is the Laplacian restricted to functions vanishing at the vicinity of the points of Y.

Point interactions 'necklaces'

A reminder: fixing the points $y_j \in Y$ the said expansions look as

$$\psi(x) = -\frac{1}{2\pi} \log|x - y_j|\, L_0(\psi, y_j) + L_1(\psi, y_j) + \mathcal{O}(|x - y_j|), \quad d = 2,$$

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$$L_1(\psi, y_j) - \alpha L_0(\psi, y_j) = 0\,, \quad \alpha \in \mathbb{R};$$

the absence of interaction corresponds to $\alpha = \infty$, we refer again to

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Theorem

The ground state of $H^N_{\alpha,\Gamma}$ is uniquely maximized by a N-regular polygon.

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More results on spectral optimization will be given in the next lecture.

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- Quantum graphs can exhibit a *nontrivial $\mathcal{PT}$-symmetry* even if the corresponding Hamiltonian is self-adjoint.
- In contrast to Faber-Krahn-type results, the ground-state energy of leaky loops and point-interaction 'necklaces' is *maximized* by configurations of the maximum symmetry